Computational Solid Mechanics

Hyperelasticity

By

John Hanna

MSc. in Computational Mechanics
Universitat Politcnica de Catalunya

$30^{th} April, 2019$

1 Given the stored elastic energy function, Cauchy stress can be calculated as follows:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \frac{\lambda}{2}\frac{\partial \epsilon_{mm}^2}{\partial \epsilon_{ij}} + \mu \frac{\partial \epsilon_{mn}\epsilon_{mn}}{\partial \epsilon_{ij}} = \lambda \epsilon_{mm}\delta_{mi}\delta_{mj} + 2\mu\delta_{mi}\delta_{nj}\epsilon_{mn} = \lambda\epsilon_{mm}\delta_{ij} + 2\mu\epsilon_{ij}$$

Which can be written in tensorial form as:

$$\boxed{\sigma = \lambda tr(\epsilon)\mathbf{I} + 2\mu\epsilon}$$

2 The condition of isotropy to be checked is

$$W(\mathbf{F}) = W(\mathbf{FQ})$$

$$W = \frac{\lambda}{2}(tr(\mathbf{E}))^2 + \mu \; tr(\mathbf{E}^2)$$

$$W(\mathbf{F}) = \frac{\lambda}{2}(tr(\frac{1}{2}(\mathbf{F^TF} - \mathbf{I})))^2 + \mu \; tr((\frac{1}{2}(\mathbf{F^TF} - \mathbf{I}))^2)$$

$$W(\mathbf{FQ}) = \frac{\lambda}{2}(tr(\frac{1}{2}(\mathbf{Q^TF^TFQ} - \mathbf{I})))^2 + \mu \; tr((\frac{1}{2}(\mathbf{Q^TF^TFQ} - \mathbf{I}))^2)$$

The used properties to check the condition are $tr(A+B) = tr(A)+tr(B)$, $tr(\alpha A) = \alpha tr(A)$, $tr(AB) = tr(BA)$, and $QQ^T = I$. Using the first two properties, the condition will be satisfied if the following two equations are true.

$$tr(\mathbf{F^TF}) = tr(\mathbf{Q^TF^TFQ}) \qquad tr(\mathbf{F^TFF^TF}) = tr(\mathbf{Q^TF^TFQQ^TF^TFQ})$$

using the property $tr(AB) = tr(BA)$ and $QQ^T = I$, the following is obtained.

$$tr(\mathbf{Q^TF^TFQ}) = tr(\mathbf{QQ^TF^TF}) = tr(\mathbf{F^TF})$$

$$tr(\mathbf{Q^TF^TFQQ^TF^TFQ}) = tr(\mathbf{Q^TF^TFF^TFQ}) = tr(\mathbf{QQ^TF^TFF^TF}) = tr(\mathbf{F^TFF^TF})$$

The two parts satisfy the condition, therefore, the model is isotropic.

3 Given the stored elastic energy function, the second Piola-Kirchhoff stress can be calculated as follows:

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} = \frac{\lambda}{2}\frac{\partial E_{mm}^2}{\partial E_{ij}} + \mu \frac{\partial E_{mn}E_{mn}}{\partial E_{ij}} = \lambda E_{mm}\delta_{mi}\delta_{mj} + 2\mu\delta_{mi}\delta_{nj}E_{mn} = \lambda E_{mm}\delta_{ij} + 2\mu E_{ij}$$

Which can be written in tensorial form as

$$\boxed{\mathbf{S} = \lambda tr(\mathbf{E})I + 2\mu\mathbf{E}}$$

4 To obtain the nominal stress, first the deformation gradient is calculated

$$\mathbf{F} = \frac{d\mathbf{x}}{d\mathbf{X}} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F^TF} - \mathbf{I}) = \frac{1}{2}\begin{bmatrix} \Lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{FS} = \lambda tr(\mathbf{E})\mathbf{F} + 2\mu\mathbf{FE} = \frac{1}{2}\lambda(\Lambda^2 - 1)\begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mu\begin{bmatrix} \Lambda^3 - \Lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_{xX} = (\frac{\lambda}{2} + \mu)(\Lambda^3 - \Lambda)$$

The curve is plotted using $\mu = 77$ and $\lambda = 115$.

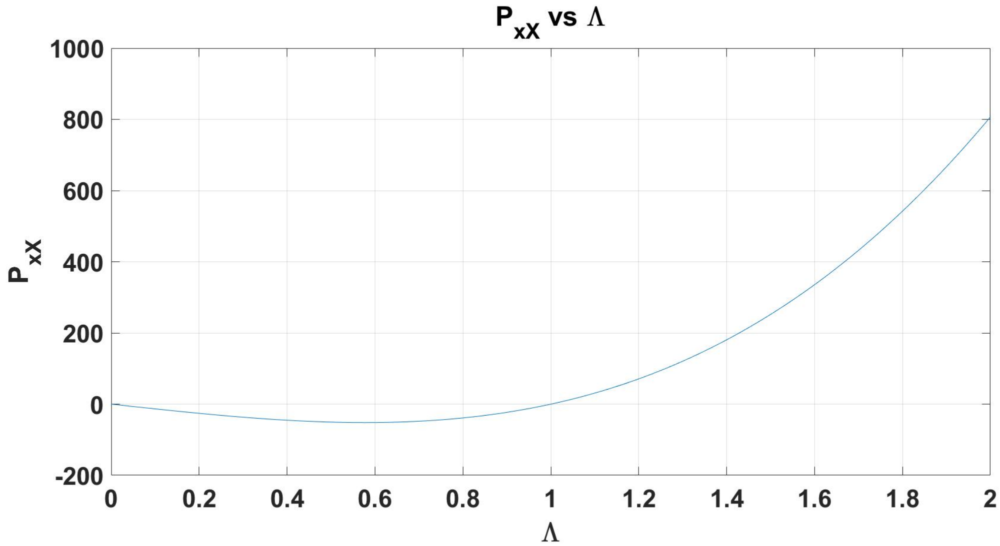

Figure 1: $P_{xX} vs \Lambda$

5 The relationship is not monotonic if the slope of the curve is zero at some point.

$$\frac{\partial P_{xX}}{\partial \Lambda} = (\frac{\lambda}{2} + \mu)(3\Lambda^2 - 1)$$

The derivative can be zero which means that the relationship is not monotonic.

$$\boxed{\Lambda_{cr} = \sqrt{\frac{1}{3}}}$$

This value doesn't depend on the elastic constants.

To check the growth condition, the Jacobin and the stored energy are calculated.

$$J = det(\mathbf{F}) = \Lambda$$

$$W = \frac{\lambda}{2}(tr(\mathbf{E}))^2 + \mu\ tr(\mathbf{E}^2) = \frac{\lambda}{8}(\Lambda^2 - 1)^2 + \frac{\mu}{4}(\Lambda^2 - 1)^2 = (\frac{\lambda}{8} + \frac{\mu}{4})(\Lambda^2 - 1)^2$$

$$\lim_{\Lambda \to 0^+} W = \lim_{\Lambda \to 0^+} (\frac{\lambda}{8} + \frac{\mu}{4})(\Lambda^2 - 1)^2 = (\frac{\lambda}{8} + \frac{\mu}{4}) \neq \infty$$

From the obtained results, it is conclude that Kirchhoff Saint-Venant material model fails after a certain compression level. If the material is compressed from its undeformed configuration, stress will increase to resist the deformation. However, when the stretch ratio reaches 0.577, the stress reaches a maximum. Further compression will lead to decrease in stress till it becomes zero when J is equal to zero.

6 To check if the new model circumvent the drawbacks of the previous model, the nominal stress is first calculated.

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} = \mathbf{F}\frac{\partial W}{\partial \mathbf{E}} = \lambda ln\ J J\mathbf{F}^{-\mathbf{T}} + 2\mu\mathbf{F}\mathbf{E}$$

$$P_{xX} = \lambda ln\Lambda + \mu(\Lambda^3 - \Lambda)$$

$$\frac{\partial P_{xX}}{\partial \Lambda} = \frac{\lambda}{\Lambda} + \mu(3\Lambda^2 - 1)$$

The obtained derivative can be zero depending on the elastic constants.

$$\lim_{\Lambda \to 0^+} W = \lim_{\Lambda \to 0^+} \frac{\lambda}{2}(ln\ \Lambda)^2 + \mu(\Lambda^2 - 1)^2 = \infty \qquad only\ \ if\ \ \lambda > 0$$

It is concluded that the modified material model circumvents the drawbacks of the previous model only with a good choice of the elastic constants.