COMPUTATIONAL SOLID MECHANICS

EXERCISE

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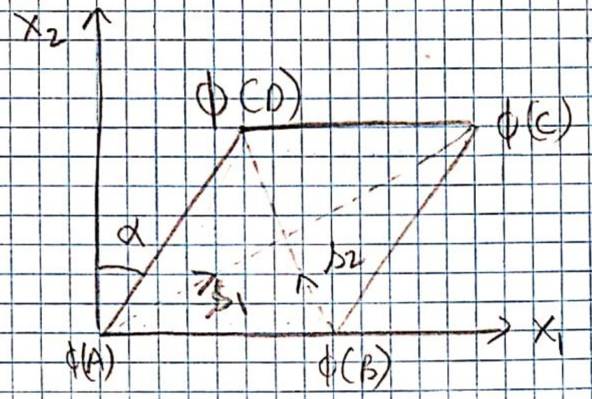
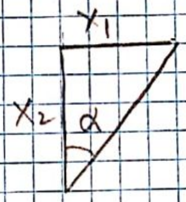
MASTERS IN COMPUTATIONAL MECHANICS

HW1_b





$X_2 = \cos\alpha$
$X_1 = \sin\alpha$

1. Deformation Mapping $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \phi \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$

$x_2 = X_2 \cos\alpha$

$x_1 = X_1 + X_2 \sin\alpha$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 + X_2 \sin\alpha \\ X_2 \cos\alpha \end{Bmatrix} \begin{matrix} \rightarrow \phi_{x_1} \\ \rightarrow \phi_{x_2} \end{matrix}$$

2. $$F = \begin{bmatrix} \dfrac{\partial \phi_{x_1}}{\partial X_1} & \dfrac{\partial \phi_{x_1}}{\partial X_2} \\ \dfrac{\partial \phi_{x_2}}{\partial X_1} & \dfrac{\partial \phi_{x_2}}{\partial X_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix}$$

Right cauchy green deformation tensor

$$C = F^T F = \begin{bmatrix} 1 & 0 \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix}$$

3. $dv = J\,dV$

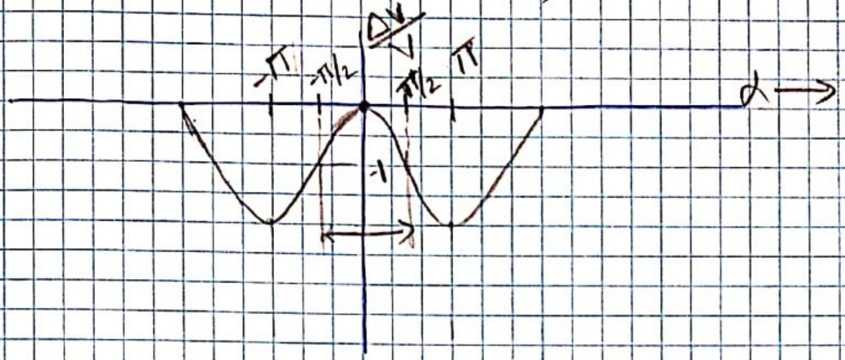
$J = \det|F|$ as J is homogeneous (given)

$v = JV = (\det F)V = (\cos\alpha)\,V$

Change in Vol. $(\Delta V) = v - V = V(\cos\alpha - 1)$

Variation of vol. as funct^n of α Plot

$$\Delta V = V(\cos\alpha - 1)$$



4. $J<0$, deformations cease to be admissible.

$J = \cos\alpha$

So, deformations are unadmissible if $\alpha > 90°$

when $\alpha = 90°$, all the ringed points are co-linear which geometrically look like Fig 3.

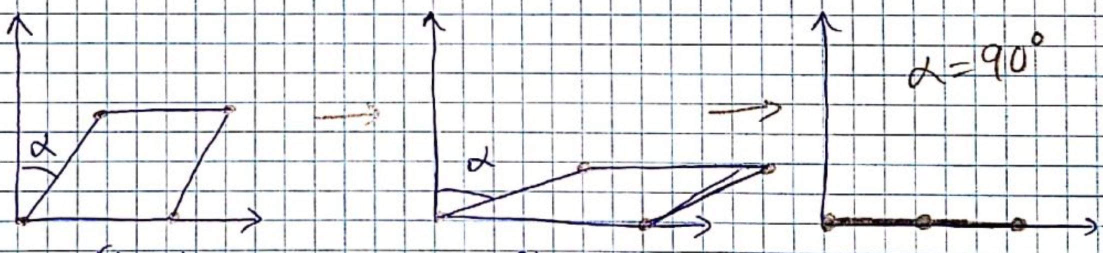


Fig 1 Fig 2 Fig 3

∴ The vol. of solid ~~dimish~~ diminishes when $\alpha = 90°$, which is impossible. So, $\alpha > 90°$ is not possible

5. $\lambda_1^2 = N_1 \cdot C \cdot N_1$; $N_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\lambda_2^2 = N_2 \cdot C \cdot N_2$; $N_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

~~Ass~~ β is the angle b/w the diagnols.

$$\lambda_1^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1+\sin\alpha \\ 1+\sin\alpha \end{bmatrix}$$

$$= \frac{1}{2} \times 2 \times [1+\sin\alpha] = (1+\sin\alpha)$$

$$\lambda_1 = \sqrt{1+\sin\alpha}$$

$$\lambda_2^2 = N_2 C N_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \times 2 \times (1 - \sin\alpha) = 1 - \sin\alpha$$

$$\lambda_2 = \sqrt{1 - \sin\alpha}$$

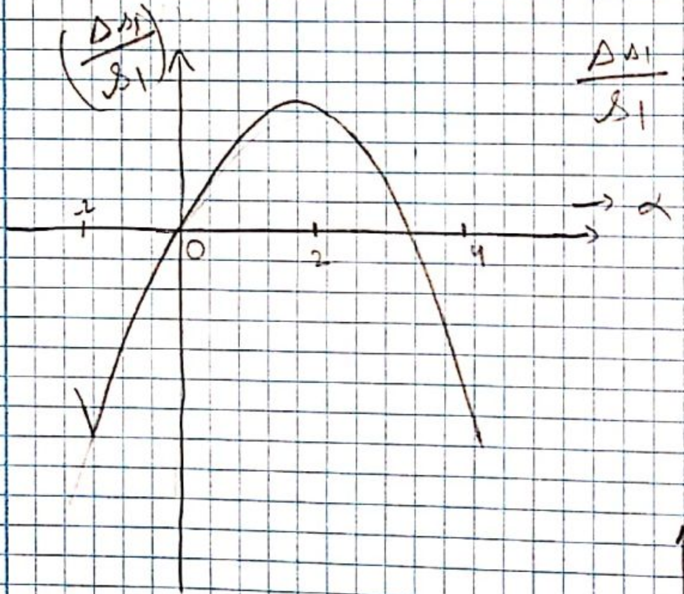
$$\lambda = \frac{ds}{dS} \Rightarrow ds = \lambda dS$$

Diagnol 1:

$$\Delta s_1 = s_1 - S_1 = \lambda_1 S_1 - S_1 = \left(\sqrt{1+\sin\alpha} - 1\right) S_1$$

Diagnol 2:

$$\Delta S_2 = s_2 - S_2 = \lambda_1 S_2 - S_2 = \left(\sqrt{1-\sin\alpha} - 1\right) S_2$$

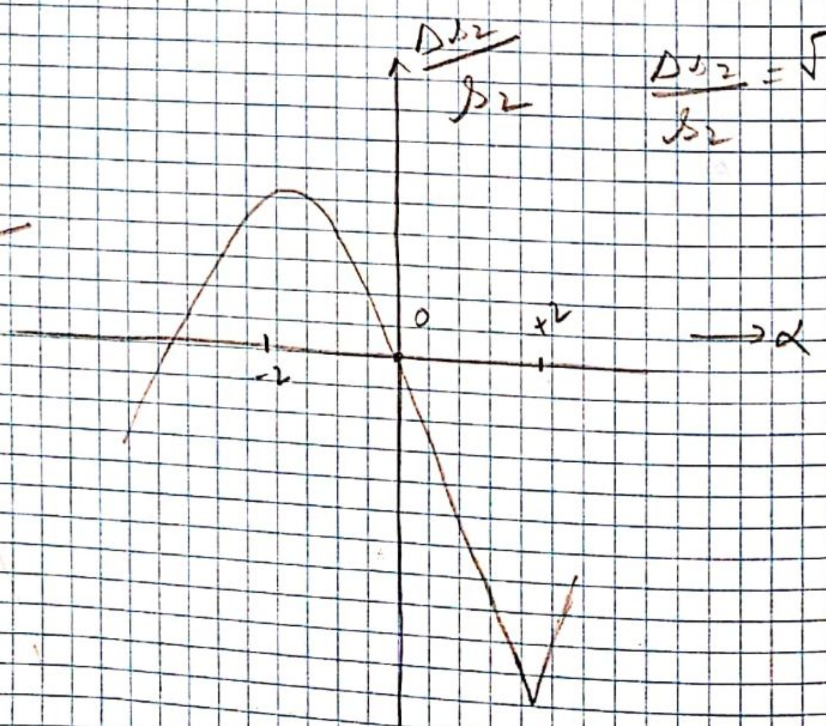


$$\frac{\Delta s_1}{S_1} = \sqrt{1+\sin\alpha} - 1$$

Diagnol 1

$$\frac{\Delta s_2}{S_2} = \sqrt{1-\sin\alpha} - 1$$

Diagnol 2



Scanned with CamScanner

β (angle b/w diagnols)

$$\cos\beta = \frac{N_1\,(1+2E)\,N_2}{\sqrt{1+2N_1 E N_1}\;\sqrt{1+2N_2 E N_2}}$$

$$C = 1 + 2E \Rightarrow E = \frac{1}{2}(C-1)$$

$$\cancel{C = \frac{1}{2}[\sin\alpha]\ 0}$$

$$N_1 = \frac{1}{\sqrt{2}}[1 \quad 1]$$

$$N_2 = \frac{1}{\sqrt{2}}[-1 \quad 1]$$

$$E = \frac{1}{2}\begin{bmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}$$

$$\sqrt{1+2N_1 \cdot E \cdot N_1} = \sqrt{1 + \frac{1}{2}[1 \quad 1]\begin{bmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \sqrt{1+\sin\alpha}$$

$$\sqrt{1+2N_2 E N_2} = \sqrt{1 + \frac{1}{2}[-1 \quad 1]\begin{bmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$= \sqrt{1-\sin\alpha}$$

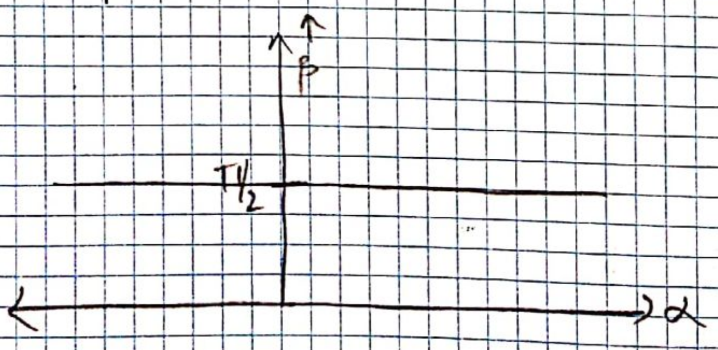
$$N_1(1+2E)N_2 = N_1 \cdot C \cdot N_2 = \frac{1}{2}[1 \quad 1]\begin{bmatrix} 1 & \sin\alpha \\ \sin\alpha & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}[1 \quad 1]\begin{bmatrix} -1+\sin\alpha \\ -\sin\alpha+1 \end{bmatrix}$$

$$= \frac{1}{2}[-1+\sin\alpha - \sin\alpha + 1] = 0$$

$\Rightarrow \cos\beta = 0$ for all values of α

$\therefore$ $\beta = 90^\circ$ at all α values.



Prakhar Rastogi

HW(1)(C)

$\phi_1 = x^1 \quad , \quad \phi_2 = x^2 \quad , \quad \phi_3 = x^3 + w(x^1, x^2)$

1.(a)
$$F = \begin{bmatrix} \frac{\partial \phi_1}{\partial x^1} & \frac{\partial \phi_1}{\partial x^2} & \frac{\partial \phi_1}{\partial x^3} \\ \frac{\partial \phi_2}{\partial x^1} & \frac{\partial \phi_2}{\partial x^2} & \frac{\partial \phi_3}{\partial x^3} \\ \frac{\partial \phi_3}{\partial x^1} & \frac{\partial \phi_3}{\partial x^2} & \frac{\partial \phi_3}{\partial x^3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 1 \end{bmatrix}$$

Right Cauchy-green deformation tensor

$$C = F^T F = \begin{bmatrix} 1 & 0 & \frac{\partial w}{\partial x^1} \\ 0 & 1 & \frac{\partial w}{\partial x^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \left(\frac{\partial w}{\partial x^1}\right)^2 & \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1} \\ \left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial w}{\partial x^2}\right) & 1 + \left(\frac{\partial w}{\partial x^2}\right)^2 & \frac{\partial w}{\partial x^2} \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 1 \end{bmatrix}$$

$J = \det(F) = 1(1-0) = 1$

(b) $dv = J\,dV$ and $J = 1$

$\Rightarrow dv = dV$; There is no change in vol. as $J = 1$

(c) $J = 1 > 0$, $\therefore$ local unpenetrability test satisfied

2.
$$A = \frac{\frac{\partial w}{\partial x^1} E_1 + \frac{\partial w}{\partial x^2} E_2}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}} = \frac{\nabla \vec{w}}{\|\nabla w\|}$$

$\rightarrow$ $\hat{a} = \frac{\vec{a}}{|a|}$ Similarly, A can be considered unit vector

$$B = \frac{-\frac{\partial w}{\partial x^2} E_1 + \frac{\partial w}{\partial x^1} E_2}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}}$$

$\therefore A \perp B$ as $\begin{bmatrix} \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} \end{bmatrix} \begin{bmatrix} -\frac{\partial w}{\partial x^2} \\ \cdots \end{bmatrix} = 0$

(b) $\lambda_A = \dfrac{ds_A}{dS_A} = \sqrt{A \cdot C \cdot A^T}$; $A = \begin{bmatrix} \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 0 \end{bmatrix}$

$$\lambda_A = \left(\frac{1}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}^{\,2}} \begin{bmatrix} \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 0 \end{bmatrix} \begin{bmatrix} 1+\left(\frac{\partial w}{\partial x^1}\right)^2 & \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1} \\ \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & 1+\left(\frac{\partial w}{\partial x^2}\right)^2 & \frac{\partial w}{\partial x^2} \\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x^1} \\ \frac{\partial w}{\partial x^2} \\ 0 \end{bmatrix} \right)^{1/2}$$

$$= \frac{1}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}^{\,2}} \left(\begin{bmatrix} \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 0 \end{bmatrix} \begin{bmatrix} \left(1+\left(\frac{\partial w}{\partial x^1}\right)^2\right)\left(\frac{\partial w}{\partial x^1}\right) + \frac{\partial w}{\partial x^1}\left(\frac{\partial w}{\partial x^2}\right)^2 \\ \left(\frac{\partial w}{\partial x^1}\right)^2\left(\frac{\partial w}{\partial x^2}\right) + \left(1+\left(\frac{\partial w}{\partial x^2}\right)^2\right)\left(\frac{\partial w}{\partial x^2}\right) \\ \left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 \end{bmatrix} \right.$$

$$= \frac{1}{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2}^{\,2}} \left[\underbrace{\left(\left(\frac{\partial w}{\partial x^1}\right)^4 + 2\left(\frac{\partial w_1}{\partial x^1}\right)^2\left(\frac{\partial w_2}{\partial x^2}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^4 \right)}_{\left(\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2\right)^2} + \left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 \right]^{1/2}$$

$$= \sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 + 1}$$

$\lambda_B = \dfrac{ds_B}{dS_B} = \sqrt{B \cdot C \cdot B^T}$; $B = \begin{bmatrix} -\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1} & 0 \end{bmatrix}$

Calculating, we get

$$\lambda_B = 1$$

Now $\quad \cos\theta = \dfrac{B \cdot C \cdot A^T}{\lambda_A \, \lambda_B}$

$$B \cdot C \cdot A^T = \begin{bmatrix} -\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1} & 0 \end{bmatrix} \begin{bmatrix} 1+\left(\frac{\partial w}{\partial x^1}\right)^2 & \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1} \\ & 1+\left(\frac{\partial w}{\partial x^2}\right)^2 & \frac{\partial w}{\partial x^2} \\ \text{sym} & & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x^1} \\ \frac{\partial w}{\partial x^2} \\ 0 \end{bmatrix}$$

$$= 0$$

$$\Rightarrow \cos\theta = 0 \quad \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

c) - $\vec{A}$ and $\vec{B}$ vectors are perpendicular to each other

- $\lambda_A = \sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 + 1} \Rightarrow \vec{A}$ change itself length

, i.e., Increment of $\vec{A}$ $ds_A > dS_A$

- $\lambda_B = 1 \Rightarrow ds_B = dS_B$

3.

$$\boxed{da = JF^{-T} dA}$$

$$J = 1 \qquad F^{-1} = \frac{(\mathrm{adj}\, F)^T}{|F|} = \begin{bmatrix} 1 & 0 & -\frac{\partial w}{\partial x^1} \\ 0 & 1 & -\frac{\partial w}{\partial x^2} \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$F^{-T} = \begin{bmatrix} 1 & 0 & -\frac{\partial w}{\partial x^1} \\ 0 & 1 & -\frac{\partial w}{\partial x^2} \\ 0 & 0 & 1 \end{bmatrix}$$

unit vector $[0 \;\; 0 \;\; 1]^T$

$$da\,\hat{n} = J F^{-T} \bar{N}\, dS$$

$$= \begin{bmatrix} -\frac{\partial w}{\partial x^1} \\ -\frac{\partial w}{\partial x^2} \\ 1 \end{bmatrix} dA$$

5.

$$\int da = \int \| J F^{-T} \bar{N} \| \, dA$$

$$= \int \underbrace{\sqrt{\left(\frac{\partial w}{\partial x^1}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2 + 1}}_{\lambda_A} \, dA$$

6. We need to derive expression for perimeter for $\phi(\partial\Omega)$

$$x^1 = x^1(s) \qquad x^2 = x^2(s)$$

$$\int dl = \int \lambda \, dL \quad ; \quad \lambda = \sqrt{1 + 2\, T \cdot E T}$$

$$E = \frac{1}{2}(C - I)$$

$$= \frac{1}{2}\begin{bmatrix} \left(\frac{\partial w}{\partial x^1}\right)^2 & \left(\frac{\partial w}{\partial x_1}\right)\left(\frac{\partial w_2}{\partial x_2}\right) & \left(\frac{\partial w}{\partial x^1}\right) \\ \left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial w}{\partial x^2}\right) & \left(\frac{\partial w}{\partial x^2}\right)^2 & \left(\frac{\partial w}{\partial x^2}\right) \\ \left(\frac{\partial w}{\partial x^1}\right) & \left(\frac{\partial w}{\partial x^2}\right) & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} \frac{\partial x^1(s)}{\partial s} & \frac{\partial x^2(s)}{\partial s} & 0 \end{bmatrix}$$

$$T \cdot E \cdot T = \frac{1}{2}\left[\left(\frac{\partial w}{\partial x^1}\right)^2\left(\frac{\partial x^1(s)}{\partial s}\right)^2 + \left(\frac{\partial w}{\partial x^2}\right)^2\left(\frac{\partial x^2(s)}{\partial s}\right)^2 + 2\left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial x^1(s)}{\partial s}\right)\left(\frac{\partial w}{\partial x^2}\right)\left(\frac{\partial x^2(s)}{\partial s}\right)\right]$$

$$= \frac{1}{2}\left(\left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial x^1(s)}{\partial s}\right) + \left(\frac{\partial w}{\partial x^2}\right)\left(\frac{\partial x^2(s)}{\partial s}\right)\right)^2$$

$$\lambda = \sqrt{1 + 2\, TET} = \sqrt{1 + \left(\left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial x^1(s)}{\partial s}\right) + \left(\frac{\partial w}{\partial x^2}\right)\left(\frac{\partial x^2(s)}{\partial s}\right)\right)^2}$$

$$\int_{\partial\Omega} da = \int_{\partial\Omega} \lambda \, dA = \int \sqrt{1 + \left(\left(\frac{\partial w}{\partial x^1}\right)\left(\frac{\partial x^1(s)}{\partial s}\right) + \left(\frac{\partial w}{\partial x^2}\right)\left(\frac{\partial x^2(s)}{\partial s}\right)\right)^2}\, dA$$