Computational Structural Mechanics and Dynamics

Assignment 3
Zahra Rajestari

Assignment 3.1

On "The Plane Stress Problem":

In isotropic elastic materials (as well as in plasticity and viscoelasticity) it is convenient to use the so-called Lamé constants λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

1. Find the inverse relations for E, ν in terms of λ, μ.
   Solution

$$\mu = \frac{E}{2(1+\nu)} \Longrightarrow E = 2\mu(1+\nu)$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\lambda = \frac{2\mu(1+\nu)\nu}{(1+\nu)(1-2\nu)} \Longrightarrow \lambda - 2\lambda\nu = 2\mu\nu \Longrightarrow 2\nu(\lambda+\mu) = \lambda \Longrightarrow \nu = \frac{\lambda}{2(\lambda+\mu)}$$

$$E = 2\mu(1+\nu) \Longrightarrow E = 2\mu(1+\frac{\lambda}{2(\lambda+\mu)}) \Longrightarrow E = \frac{\mu(3\lambda+2\mu)}{\mu+\lambda}$$

2. Express the elastic matrix for plane stress and plane strain cases in terms of λ, μ.
   Solution
   Elastic matrix for plane stress:

$$\frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Which in terms of μ and λ can be written as:

$$\frac{4\mu(2\mu+3\lambda)(\mu+\lambda)}{4(\mu+\lambda)^2-\lambda^2}\begin{bmatrix} 1 & \lambda/2(\mu+\lambda) & 0 \\ \lambda/2(\mu+\lambda) & 1 & 0 \\ 0 & 0 & 1/2-\lambda/2(\mu+\lambda) \end{bmatrix}$$

Elastic matrix for plane strain:

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}$$

Substituting the equations obtained from the previous problem for E and ν yields:

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}(1-\frac{\lambda}{2(\lambda+\mu)})}{(1+\frac{\lambda}{2(\lambda+\mu)})(1-\frac{2\lambda}{2(\lambda+\mu)})} = 2\mu+\lambda$$

$$\frac{\nu}{1-\nu} = \frac{\frac{\lambda}{2(\lambda+\mu)}}{1-\frac{\lambda}{2(\lambda+\mu)}} = \frac{\lambda}{2\mu+\lambda}$$

$$\frac{1-2\nu}{2(1-\nu)} = \frac{1-\frac{\lambda}{\lambda+\mu}}{2(1-\frac{\lambda}{2(\lambda+\mu)})} = \frac{\mu}{2\mu+\lambda}$$

Therefore, the Elastic matrix for plane strain can be written in terms of λ and μ as:

$$\begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

3. Split the stress-strain matrix E for plane strain as

$$\mathbf{E} = \mathbf{E}_\lambda + \mathbf{E}_\mu$$

in which E_μ and E_λ contain only μ and λ, respectively.
This is the Lamé λ,μ splitting of the plane strain constitutive equations, which leads to the so-called B-bar formulation of near-incompressible finite elements.

<u>Solution</u>

$$E_\lambda = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \; E_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

4. Express E_λ and E_μ also in terms of E and ν.
<u>Solution</u>

$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; E_\mu = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

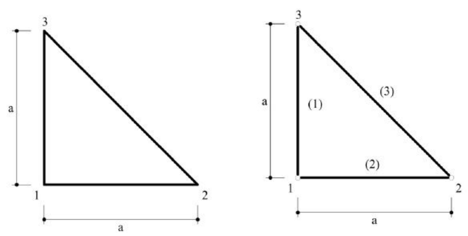
Assignment 3.2

On "The 3-node Plane Stress Triangle":

Consider a plane triangular domain of thickness h, with horizontal and vertical edges of length a. Let us consider for simplicity $a = 1$, $h = 1$. The material parameters are E, ν. Initially ν is set to zero. Two discrete structural models are considered as depicted in the figure:

(a) A plane linear Turner triangle with the same dimensions.

(b) A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.



1. Calculate the stiffness matrices K_{tri} and K_{bar} for both discrete models.

2. Is there any set of values for the cross sections A_1=A_2 and A_3 to make both stiffness matrix equivalent: K_{tri}=K_{bar}? If not, which are the values that make them more similar?

3. Why these two stiffness matrices are not equal?. Find a physical explanation.

4. Consider now $\nu \neq 0$ and extract some conclusions.

Solution

For plane stress we have:

$$K^{(e)} = \int_{\Omega^{(e)}} h\mathbf{B^T E B}\, d\Omega^{(e)}$$

where B is:

$$\begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \cdots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \cdots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \cdots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix}$$

The shape functions of the triangular element can be written in terms of the natural coordinates as the following:

$$N_1 = 1 - \xi - \eta$$

$$N_2 = \xi$$

$$N_3 = \eta$$

We have:

$$
\begin{aligned}
\frac{\partial \xi}{\partial x} &= 1 \\
\frac{\partial \eta}{\partial x} &= 0 \\
\frac{\partial \xi}{\partial y} &= 0 \\
\frac{\partial \eta}{\partial y} &= 1
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\frac{\partial N_1}{\partial x} &= \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial x} = -1 \\
\frac{\partial N_1}{\partial y} &= \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial y} = -1 \\
\frac{\partial N_2}{\partial x} &= \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x} = 1 \\
\frac{\partial N_2}{\partial y} &= \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial y} = 0 \\
\frac{\partial N_3}{\partial x} &= \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial x} = 0 \\
\frac{\partial N_3}{\partial y} &= \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial y} = 1
\end{aligned}
$$

Which yields:

$$
B = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}
$$

Elastic matrix for plane stress is:

$$
\frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}
$$

So the element stiffness matrix is computed as the following:

$$
K_{tri} = \frac{1}{2}B^T E B = \frac{1}{2}\frac{E}{(1-\nu^2)}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}
$$

$$
K_{tri} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 3/4-\nu/4 & \nu/4+1/4 & -1/2 & \nu/4-1/4 & \nu/4-1/4 & -\nu/2 \\ \nu/4+1/4 & 3/4-\nu/4 & -\nu/2 & \nu/4-1/4 & \nu/4-1/4 & -1/2 \\ -1/2 & -\nu/2 & 1/2 & 0 & 0 & \nu/2 \\ \nu/4-1/4 & \nu/4-1/4 & 0 & 1/4-\nu/4 & 1/4-\nu/4 & 0 \\ \nu/4-1/4 & \nu/4-1/4 & 0 & 1/4-\nu/4 & 1/4-\nu/4 & 0 \\ -\nu/2 & -1/2 & \nu/2 & 0 & 0 & 1/2 \end{bmatrix}
$$

Considering $\nu = 0$, we have:

$$K_{tri} = E \begin{bmatrix} 3/4 & 1/4 & -1/2 & -1/4 & -1/4 & 0 \\ 1/4 & 3/4 & 0 & -1/4 & -1/4 & -1/2 \\ -1/2 & 0 & 1/2 & 0 & 0 & 0 \\ -1/4 & -1/4 & 0 & 1/4 & 1/4 & 0 \\ -1/4 & -1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

For K_{bar} which consists of three bar elements with $A_1 = A_2 = A$ and A_3, we have:

$$K^{(1)} = E \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & A \end{bmatrix}$$

$$K^{(2)} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(3)} = \frac{E}{2\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & -A_3 & -A_3 & A_3 \\ 0 & 0 & -A_3 & A_3 & A_3 & -A_3 \\ 0 & 0 & -A_3 & A_3 & A_3 & -A_3 \\ 0 & 0 & A_3 & -A_3 & -A_3 & A_3 \end{bmatrix}$$

$$K_{bar} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A + A_3/2\sqrt{2} & -A_3/2\sqrt{2} & -A_3/2\sqrt{2} & A_3/2\sqrt{2} \\ 0 & -A & -A_3/2\sqrt{2} & A_3/2\sqrt{2} & A_3/2\sqrt{2} & -A_3/2\sqrt{2} \\ -A & 0 & -A_3/2\sqrt{2} & A_3/2\sqrt{2} & A_3/2\sqrt{2} & -A_3/2\sqrt{2} \\ 0 & -A & A_3/2\sqrt{2} & -A_3/2\sqrt{2} & -A_3/2\sqrt{2} & A + A_3/2\sqrt{2} \end{bmatrix}$$

We have two different elements. The bar element is one dimensional while the triangular element is two dimensional. The triangular element is able to tolerate shear stress. From my view point, the main difference is because of the area of the elements. In the triangular element we have the area of the element to be equal to the area of a triangle. In bar element the area is not the same as a triangle and it is the area of each bar. The bar element has more zeros than triangular which means that some of the degrees of freedom do not have influence on the result.