CSMD

Assignment 4

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1. $$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \underset{\sim}{u} \doteq \underset{\sim}{N}\,\underset{\sim}{u}$$

$$\underset{\sim}{u} = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{bmatrix} ; \qquad \underset{\sim}{e} = \begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \\ 2e_{rz} \end{bmatrix} = \underset{\sim}{B}\,\underset{\sim}{u} ;$$

$$\underset{\sim}{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 \\ \frac{N_i}{r} & 0 \\ 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} \end{bmatrix} ;$$

We choose the shape function like below:

$(i=1,2,3)$

$$N_i = \frac{1}{2A}(a_i + b_i r + c_i z) ;$$

① $$\begin{cases} a_1 = r_2 z_3 - r_3 z_2 = ab \\ b_1 = z_2 - z_3 = -b \\ c_1 = 0 \end{cases}$$

② $$\begin{cases} a_2 = r_3 z_1 - r_1 z_3 = 0 \\ b_2 = z_3 - z_1 = b \\ c_2 = r_1 - r_3 = -a \end{cases}$$

③ $$\begin{cases} a_3 = r_1 z_2 - r_2 z_1 = 0 \\ b_3 = z_1 - z_2 = 0 \\ c_3 = r_2 - r_1 = a \end{cases}$$

So we get:

$$\underset{\sim}{B}_r = \frac{1}{2A}\left[\begin{array}{cc|cc|cc} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ f_1 & 0 & f_2 & 0 & f_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{array}\right] ; \quad \begin{cases} f_1 = \dfrac{a_1 + b_1 r + c_1 z}{r} = \dfrac{ab - br}{r} \\ f_2 = \dfrac{br - az}{r} \\ f_3 = a\dfrac{z}{r} \end{cases}$$

$$\Rightarrow \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} = \frac{1}{4A^2}\begin{bmatrix} b^2+f_1^2 & 0 & -b^2+f_1f_2 & 0 & f_1f_3 & 0 \\ 0 & b^2/2 & ab/2 & -b^2/2 & -ab/2 & 0 \\ -b^2+f_1f_2 & ab/2 & b^2+f_2^2+a^2/2 & -ab/2 & f_2f_3-a^2/2 & 0 \\ 0 & -b^2/2 & -ab/2 & a^2+b^2/2 & ab/2 & -a^2 \\ f_1f_3 & -ab/2 & f_2f_3-a^2/2 & ab/2 & f_3^2+a^2/2 & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}$$

(This matrix is symmetric)

$$\Rightarrow K^e = \iint \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B}\, r\, dr\, dz$$

$$= \begin{bmatrix} \frac{2}{3}b & 0 & -b/4 & 0 & b/12 & 0 \\ & b/6 & a/6 & -b/6 & a/6 & 0 \\ & & 4b/9 + a^2/6b & -a/6 & b/18 - a^2/6b & 0 \\ & & & a^2/3b + b/6 & a/6 & -a^2/3b \\ & & & & b/9 & 0 \\ \text{Symmetry} & & & & & a^2/3b \end{bmatrix}$$

We can ~~fond~~ find that the 2, 4, 6 rows and columns' sum ~~is~~ zero. But the 1, 3, 5 rows and columns don't follow this.

Reason

① for rows 2, 4, 6.

$\underset{\sim}{f}^e = \underset{\sim}{k}^e \underset{\sim}{u}^e$; the sum of 2, 4, 6 rows is zero. this situation can only happen when $u_1^e = u_2^e = u_3^e = w_1^e = w_2^e = w_3^e$. which means that the element only has rigid motion. The element is actually in 3-D, not 2-D like simplified. So the ~~rigid~~ r-direction motion means that the element has extended on r-direction. So it needs forces on r direction. This is the reason why 1, 3, 5 rows' sum is not zero. But on Z-direction, the situation is different. the element only has rigid displacement which doesn't need force and the $f_{z_1} = f_{z_2} = f_{z_3} = 0$ under this condition. so the sum of 2, 4, 6 becomes zero.

② for columns 2, 4, 6

Because the $\underset{\sim}{k}^e$ matrix is symmetric, so the ~~columns~~ 2, 4, 6 columns' sum is also zero.

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$$\underset{\sim}{f}^e = \iint \underset{\sim}{N}^T \underset{\sim}{b} \cdot r \cdot dr dz \quad ; \quad \underset{\sim}{b} = [0, -g]^T$$

$$= \iint \begin{bmatrix} 0 \\ -N_1 g \\ 0 \\ -N_2 g \\ 0 \\ -N_3 g \end{bmatrix} \cdot r \cdot dr dz$$

$$= ab \begin{bmatrix} 0 \\ a/4 - ab/3 \\ 0 \\ -a/8 \\ 0 \\ -a/8 \end{bmatrix}$$