# Assignment 4

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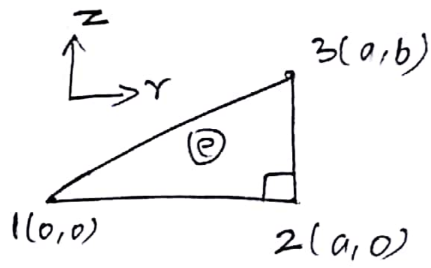
① => Given

the co-ordinates of axisymmetric triangle.

$(r_1, z_1) = (0,0)$ ; $(r_2, z_2) = (a, 0)$

$(r_3, z_3) = (a, b)$




stress-strain matrix

$$\underline{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & a \\ 0 & 0 & b \end{bmatrix}$$

$$2A = ab$$

②

The isoparametric representation for triangular element is given by

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r_1} & u_{r_2} & u_{r_3} \\ u_{z_1} & u_{z_2} & u_{z_3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

$$N_1 = \zeta_1 \quad N_2 = \zeta_2 \quad N_3 = \zeta_3$$

where, as per the element geometry

$$\begin{bmatrix} 1 \\ r \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - r_3 z_2 & z_2 - z_3 & r_3 - r_2 \\ r_3 z_1 - r_1 z_3 & z_3 - z_1 & r_1 - r_3 \\ r_1 z_2 - r_2 z_1 & z_1 - z_2 & r_2 - r_1 \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{1}{ab} \begin{bmatrix} ab & -b & 0 \\ 0 & b & -a \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 \\ r \\ z \end{bmatrix}$$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{r}{a} \\ \frac{r}{a} - \frac{z}{b} \\ \frac{z}{b} \end{bmatrix}$$

For the element stiffness matrix

$$K^e = \int_{\Omega^e} r\, B^{eT}\, \underline{E}\, B^e\, d\Omega$$

We know

$$B^e = \frac{1}{2A}\begin{bmatrix} z_{23} & 0 & z_{31} & 0 & z_{12} & 0 \\ 0 & r_{32} & 0 & r_{13} & 0 & r_{21} \\ \frac{2AN_1}{r} & 0 & \frac{2AN_2}{r} & 0 & \frac{2AN_3}{r} & 0 \\ r_{32} & z_{23} & r_{13} & z_{31} & r_{21} & z_{12} \end{bmatrix}$$

$$B^e = \frac{1}{2A}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{2AN_1}{r} & 0 & \frac{2AN_2}{r} & 0 & \frac{2AN_3}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$\Rightarrow K^e = \int_{\Omega^e} r \frac{1}{2A}\begin{bmatrix} -b & 0 & \frac{2AN_1}{r} & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & \frac{2AN_2}{r} & -a \\ 0 & -a & 0 & b \\ 0 & 0 & \frac{2AN_3}{r} & a \\ 0 & a & 0 & 0 \end{bmatrix} E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \frac{1}{2A}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{2AN_1}{r} & 0 & \frac{2AN_2}{r} & 0 & \frac{2AN_3}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$K^e = \frac{E}{4A^2}\int_{\Omega^e} \begin{bmatrix} b^2r + \frac{4A^2 N_1 N_1}{r} & 0 & -b^2r + \frac{4A^2 N_1 N_2}{r} & 0 & \frac{4A^2 N_1 N_2}{r} & 0 \\ 0 & \frac{b^2 r}{2} & \frac{abr}{2} & -\frac{b^2 r}{2} & \frac{-abr}{2} & 0 \\ -b^2r + \frac{4A^2 N_1 N_2}{r} & \frac{-abr}{2} & b^2 r + \frac{4A^2 N_2 N_2}{r} + \frac{a^2 r}{2} & \frac{-abr}{2} & -\frac{a^2 r}{2} + \frac{4A^2 N_2 N_3}{r} & 0 \\ 0 & -\frac{b^2 r}{2} & -\frac{abr}{2} & a^2 r + \frac{b^2 r}{2} & \frac{abr}{2} & -a^2 r \\ \frac{4A^2 N_1 N_3}{r} & -\frac{abr}{2} & \frac{4A^2 N_2 N_3}{r} - \frac{a^2 r}{2} & \frac{abr}{2} & \frac{a^2 r}{2} + \frac{4A^2 N_3 N_3}{r} & 0 \\ 0 & 0 & 0 & -a^2 r & 0 & a^2 r \end{bmatrix} d\Omega$$

After integrating over the domain, we get elemental stiffness matrix.

$$K^e = E \begin{bmatrix} \frac{2b}{3} & 0 & -\frac{b}{4} & 0 & \frac{ab}{24} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & \frac{-a}{6} & 0 \\ -\frac{b}{4} & \frac{a}{6} & \frac{7b}{36} + \frac{a^2}{6b} & -\frac{a}{6} & \frac{b}{18} - \frac{a^2}{6b} & 0 \\ 0 & -\frac{b}{6} & -\frac{a}{6} & \frac{b}{6} + \frac{a^2}{3b} & \frac{a}{6} & -\frac{a^2}{3b} \\ \frac{ab}{24} & -\frac{a}{6} & \frac{b}{18} - \frac{a^2}{6b} & \frac{a}{6} & \frac{a^2}{6b} + \frac{b}{9} & 0 \\ 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

(2) ⇒ By referencing the elemental matrix

Sum of rows 2, 4 and 6 gives.

$$= \begin{bmatrix} 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{b}{6} & -\frac{a}{6} & \frac{b}{6}+\frac{a^2}{3b} & \frac{a}{6} & -\frac{a^2}{3b} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ Since rigid body motion can be obtained in Z direction

By referencing the elemental matrix

Sum of rows of 1, 3 and 5 gives

$$= \begin{bmatrix} \frac{2b}{3} & 0 & -\frac{b}{4} & 0 & \frac{ab}{24} & 0 \end{bmatrix} + \begin{bmatrix} -\frac{b}{4} & \frac{a}{6} & \left(\frac{7b}{36}+\frac{a^2}{6b}\right) & -\frac{a}{6} & \left(\frac{b}{18}-\frac{a^2}{6b}\right) & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{ab}{24} & -\frac{a}{6} & \left(-\frac{a^2}{6b}+\frac{b}{18}\right) & \frac{a}{6} & \left(\frac{a^2}{6b}+\frac{b}{9}\right) & 0 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ Since rigid body motion cannot be obtained in the r-direction

①

③ ⇒ Given

gravity forces

$b = [0, -g]^T$ per unit mass

We know

$$f_{ext}^{(e)} = \int_{\Omega^e} [N^T b r] \, d\Omega;$$

$$N^e = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix}$$

$b = [0 \;\; -\rho g]^T$ [∵ per unit volume]

We have,

$$\begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} = \begin{bmatrix} 1 - \frac{r}{a} \\ \frac{r}{a} - \frac{z}{b} \\ \frac{z}{b} \end{bmatrix}$$

$$\Rightarrow \quad f_{ext}^{(e)} = \int_{\Omega^e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} r \, d\Omega;$$

$$= \int_{\Omega^e} \begin{bmatrix} 0 \\ -\rho g N_1 r \\ 0 \\ -\rho g N_2 r \\ 0 \\ -\rho g N_3 r \end{bmatrix} d\Omega = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

②

$\Rightarrow f_1 = 0$

$$\Rightarrow f_2 = \int_{\Omega_e} -\rho g r N_1 \, d\Omega$$

$$= -\rho g \int_{r=0}^{r=a} \left[ \int_{z=0}^{z=\frac{br}{a}} \left[ \left( \frac{ar - r^2}{a} \right) \right] dz \right] dr \cdot \quad \left[ \because N_1 = 1 - \frac{r}{a} \right]$$

$$= -\rho g \int_{r=0}^{r=a} \left[ \frac{abr^2 - br^3}{a^2} \right] dr$$

$$= -\frac{\rho g a^2 b}{12}$$

$\Rightarrow f_3 = 0$

$$\Rightarrow f_4 = \int_{\Omega_e} -\rho g r N_2 \, d\Omega$$

$$f_4 = -\rho g \int_{r=0}^{r=a} \left[ \int_{z=0}^{z=\frac{br}{a}} \left[ \left( \frac{r^2 b - zar}{ab} \right) \right] dz \right] dr \quad \left[ \because N_2 = \frac{r}{a} - \frac{z}{b} \right.$$

$$= -\rho g \int_{r=0}^{r=0} \left[ \frac{br^3}{2a^2} \right] dr$$

$$f_4 = -\frac{\rho g a^2 b}{8}$$

$\Rightarrow f_5 = 0$

③

$$\Rightarrow f_6 = \int_{\Omega^e} -\rho g r N_3 \, d\Omega$$

$$f_6 = -\rho g \int_{r=0}^{r=a} \left[ \int_{z=0}^{z=\frac{br}{a}} \left[ \left( \frac{zr}{b} \right) \right] dz \right] dr \quad \left[ N_3 = \frac{z}{b} \right.$$

$$= -\rho g \int_{r=0}^{r=a} \left[ \frac{br^3}{2a^2} \right] dr$$

$$f_6 = -\frac{\rho g a^2 b}{8}$$

$$f_{ext}^{(e)} = \begin{bmatrix} 0 \\ -\frac{\rho g a^2 b}{12} \\ 0 \\ -\frac{\rho g a^2 b}{8} \\ 0 \\ -\frac{\rho g a^2 b}{8} \end{bmatrix}$$

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