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Computational structural mechanics and dynamics

assignment #1

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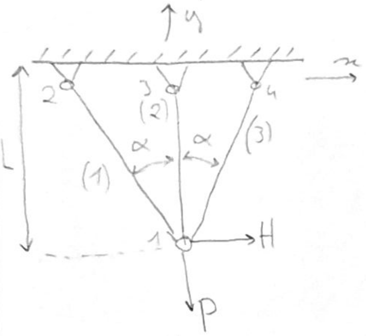
* The element stiffness matrix including the transformation takes the following form

$$K^e = \frac{E^e A^e}{l^e}\begin{bmatrix} C^2 & SC & -C^2 & -SC \\ & S^2 & -SC & -S^2 \\ & & C^2 & SC \\ \text{sym.} & & & S^2 \end{bmatrix}$$

; where E, A are the same for all elements $\Rightarrow$ in compact form:

$$K^e = \frac{E^e A^e}{l^e}\begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}$$

* The truss problem is as follows: $\Rightarrow$ hence the connectivity matrix takes the form:



$$T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

* The stiffness matrix for each element could be formulated taking into consideration the angle of each element:

(2): angle $= 90^\circ$;

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix}$$

(3): angle $= 90^\circ - \alpha$;

$\cos(90-\alpha) = \sin\alpha$

$\sin(90-\alpha) = \cos\alpha$

Where $l^e = \frac{L}{\cos\alpha}$

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} S^2C & SC^2 & -S^2C & -SC^2 \\ SC^2 & C^3 & -SC^2 & -C^3 \\ -S^2C & -SC^2 & S^2C & SC^2 \\ -SC^2 & -C^3 & SC^2 & C^3 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} \end{bmatrix}$$

(1): angle $= 90^\circ + \alpha$;

$\cos(90+\alpha) = -\sin\alpha$

$\sin(90+\alpha) = \cos\alpha$

Where $l^e = \frac{L}{\cos\alpha}$

$$K^{(1)} = \frac{EA}{L}\begin{bmatrix} S^2C & -SC^2 & -S^2C & SC^2 \\ -SC^2 & C^3 & SC^2 & -C^3 \\ -S^2C & SC^2 & S^2C & -SC^2 \\ SC^2 & -C^3 & -SC^2 & C^3 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{bmatrix}$$

* The Global stiffness matrix is obtained by combining the elemental stiffness matrices as follows:

$$K^G = \frac{AE}{L}\begin{bmatrix} K_{11}^{(1)} + K_{11}^{(2)} + K_{11}^{(3)} & K_{12}^{(1)} & K_{12}^{(2)} & K_{12}^{(3)} \\ K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 \\ K_{21}^{(2)} & 0 & K_{22}^{(2)} & 0 \\ K_{21}^{(3)} & 0 & 0 & K_{22}^{(3)} \end{bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} 2s^2c & 0 & -s^2c & sc^2 & 0 & 0 & -s^2c & -sc^2 \\ & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^3 \\ & & s^2c & -sc^2 & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & \text{sym.} & & & & 1 & 0 & 0 \\ & & & & & & s^2c & sc^2 \\ & & & & & & & c^3 \end{bmatrix}$$

* The 5th row and column are associated with the lateral component of the stiffness of element (2). Trusses are designed to carry loads only axially. Hence the resistance to motion or stiffness in the lateral direction is non-existent hence the zero terms. There is no relation between the stiffness and the displacement at the 3rd node in the x-direction.

b) The structure is fixed at nodes 2, 3 and 4 hence $u_{x2} = u_{x3} = u_{x4} = u_{y2} = u_{y3} = u_{y4} = 0$

The problem matrices become:

$$\frac{EA}{L}\begin{bmatrix} 2s^2c & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix}$$

c) $$\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \frac{EA}{(2s^2c)(1+2c^3)L}\begin{bmatrix} 1+2c^3 & 0 \\ 0 & 2s^2c \end{bmatrix}\begin{bmatrix} H \\ -P \end{bmatrix}$$

$$u_{x1} = \frac{EA\cancel{(1+2c^3)}H}{(2s^2c)\cancel{(1+2c^3)}L} \quad ; \quad u_{y1} = \frac{-EA(2s^2c)P}{(2s^2c)(1+2c^3)L}$$

$$\lim_{\alpha \to 0} u_{x1} = \frac{EAH}{0} = \infty \qquad \lim_{\alpha \to 0} u_{y1} = \frac{-EAP}{3L}$$

In this case we have three coinciding trusses in the vertical direction. Hence the axial disp is for 3 trusses and lateral disp is ∞ because the truss member has no lateral stiffness or resistance to motion

$$\lim_{\alpha \to \pi/2} u_{x1} = \frac{EAH}{0} = \infty \qquad \lim_{\alpha \to \pi/2} u_{y1} = \frac{-EAP}{0} = -\infty$$

The solution in this case has no physical meaning because we would have infinitely long elements (1) and (3) which makes no physical sense d)

For element (3):

$$\bar{u}^{(3)} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} EAH/2s^2cL \\ -EAP/(1+2c^3)L \\ 0 \\ 0 \end{bmatrix}$$

$$d^{(3)} = \cancel{u_{x4}}^{\,0} - u_{x1} = \frac{EA}{L}\left(\frac{-H}{2sc} + \frac{cP}{1+2c^3}\right)$$

$$F^{(3)} = \frac{EAc}{L} d^{(3)} = \left(\frac{EA}{L}\right)^2 \left(\frac{c^2P}{1+2c^3} - \frac{H}{2s}\right)$$

For element (1):

$$\bar{u}^{(1)} = \begin{bmatrix} -s & -c & 0 & 0 \\ c & -s & 0 & 0 \\ 0 & 0 & -s & -c \\ 0 & 0 & c & -s \end{bmatrix} \begin{bmatrix} EAH/2s^2cL \\ -EAP/(1+2c^3)L \\ 0 \\ 0 \end{bmatrix} \Rightarrow F^{(1)} = \left(\frac{EA}{L}\right)^2 \left(\frac{H}{2s} - \frac{c^2P}{1+2c^3}\right)$$

For element (2):

$$F^{(2)} = \left(\frac{EA}{L}\right)^2 \left(\frac{-Hc}{2s^2} + \frac{scP}{1+2c^3}\right)$$

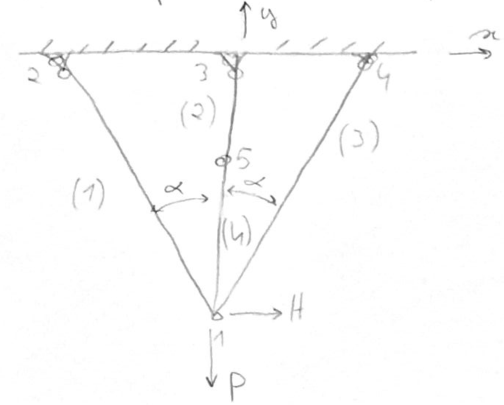
$$* \lim_{\alpha \to 0} F^{(1)} = \lim_{\alpha \to 0} \left(\frac{EA}{L}\right)^2 \left(\frac{H}{2s} - \frac{c^2P}{1+2c^3}\right) = \frac{H}{0} - \frac{P}{3} = \infty$$

$$\lim_{\alpha \to 0} F^{(3)} = \lim_{\alpha \to 0} \left(\frac{EA}{L}\right)^2 \left(\frac{c^2P}{1+2c^3} - \frac{H}{2s}\right) = \frac{P}{3} - \frac{H}{0} = \infty$$

They blow up because in the case where $\alpha \to 0$ the problem is reduced to three coinciding trusses in the vertical axis. Where $H \neq 0$ we have a lateral load to this structure. The truss member is designed to carry axial loads only hence the existance of a lateral load could not be modeled hence the math "blows up"

Assignment 2

The new problem takes the form:



connectivity matrix

$$T = \begin{bmatrix} 1 & 5 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

The Global stiffness matrix takes the form:

$$K^G = \frac{AE}{L}\begin{bmatrix} K_{11}^{(1)} + K_{11}^{(4)} + K_{11}^{(3)} & K_{12}^{(1)} & K_{12}^{(4)} & K_{12}^{(3)} & K_{12}^{(4)} \\ K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 & 0 \\ K_{21}^{(5)} & 0 & K_{22}^{(5)} & 0 & 0 \\ K_{21}^{(3)} & 0 & 0 & K_{22}^{(3)} & 0 \\ K_{21}^{(4)} & 0 & 0 & 0 & K_{22}^{(4)} \end{bmatrix}$$

The reduced system takes the form:

$$\frac{AE}{L}\begin{bmatrix} K_{11}^{(1)} + K_{11}^{(4)} + K_{11}^{(3)} & K_{12}^{(4)} \\ K_{21}^{(4)} & K_{22}^{(4)} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x5} \\ u_{y5} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Where } K^G_{reduced} = \frac{AE}{L}\begin{bmatrix} 2s^2 & 0 & 0 & 0 \\ 0 & 1+2c^3 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

It could be shown that the K^G reduced matrix is not invertible as it has a row of zeros (the determinant in this case is zero). Hence a unique solution does not exist for the prescribed system. This is characteristic of a rigid body motion. By adding an extra degree of freedom, the system became ill-conditioned hence requires more BC or loads to make the system computationally stable/solvable.