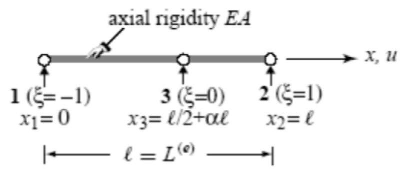
HOMEWORK 4

Assignment 4.1: Isoparametric representation



1) Get the Jacobian (J) and show that

- **If -¼ < α < ¼, J>0 over the whole element -1 ≤ ξ ≤ 1.**
- **If α = 0, J = L/2 over the element.**

The position of the nodes are written as,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

and its derivate define the Jacobian,

$$J = \frac{dx}{d\xi} = \frac{dN_1}{d\xi} x_1 + \frac{dN_2}{d\xi} x_2 + \frac{dN_3}{d\xi} x_3$$

The shape functions are defined as

$$N_1 = \frac{\xi(-1+\xi)}{2} \rightarrow \frac{dN_1}{d\xi} = \frac{2\xi - 1}{2}$$

$$N_2 = \frac{\xi(1+\xi)}{2} \rightarrow \frac{dN_2}{d\xi} = \frac{2\xi + 1}{2}$$

$$N_3 = (1 - \xi^2) \rightarrow \frac{dN_3}{d\xi} = -2\xi$$

Substituting the values of the shape function in the Jacobian

$$J = \frac{dx}{d\xi} = \frac{2\xi - 1}{2} 0 + \frac{2\xi + 1}{2} L + -2\xi \left(\frac{L}{2} + \alpha L\right) = L\left(\frac{1}{2} - 2\alpha\xi\right)$$

- **<u>If -¼ < α < ¼, J>0 over the whole element -1 ≤ ξ ≤ 1.</u>**

The Jacobian is positive when the expression in parentheses is positive because the length of the element is always positive. Therefore, it should be studied how each parameter affect to the expression sign.

$$J > 0 \rightarrow \frac{1}{2} - 2\alpha\xi > 0$$

Firstly, it will be assumed that ξ is negative (-1< ξ <0) and will be analyzed the sign of the expression in parentheses depending on the values of α:

$$If\ \xi < 0 \quad \rightarrow \quad \frac{1}{2} + 2\alpha\xi$$

- Assuming α ≥ 0: The Jacobian would be always greater than zero for any positive α because the sum of both terms will be positive.
$$\frac{1}{2} + 2\alpha\xi > 0 \rightarrow \alpha > 0$$
- Assuming α < 0: In this case the second term is negative and at a certain value the sum of both terms would be negative.
$$\frac{1}{2} - 2\alpha\xi > 0 \rightarrow \alpha > -\frac{1}{4\xi} \xrightarrow{\xi=1} \alpha > -\frac{1}{4}$$
Therefore, the Jacobian would be positive for values of α greater than -¼.

Lastly, the same process will be repeated for ξ > 0:

$$If\ \xi > 0 \quad \rightarrow \quad \frac{1}{2} - 2\alpha\xi$$

- Assuming α > 0: The Jacobian would be positive only for values of α smaller than ¼.
$$\frac{1}{2} - 2\alpha\xi > 0 \rightarrow \alpha < \frac{1}{4}$$
- Assuming α ≤ 0: The Jacobian would be positive always.
$$\frac{1}{2} + 2\alpha\xi > 0 \rightarrow \alpha \leq 0$$

Takin into account both restriction α must be less than 1/4 .

In conclusion, the Jacobian would be positive for values of α between -¼ and ¼.

$$J > 0 \qquad \forall\ \alpha\ \in \left[-\frac{1}{4}, \frac{1}{4}\right]$$

- **If α = 0, J = L/2 over the element.**

Substituting the value of α into the Jacobian:

$$J(\alpha = 0) = \frac{dx}{d\xi} = L\left(\frac{1}{2} - 2\alpha\xi\right) = \frac{L}{2}$$

2) Obtain the 1x3 strain displacement matrix (B):

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi} = \frac{1}{J}\begin{bmatrix} \frac{dN_1}{d\xi} \\ \frac{dN_2}{d\xi} \\ \frac{dN_3}{d\xi} \end{bmatrix} = \frac{1}{L\left(\frac{1}{2} - 2\alpha\xi\right)}\begin{bmatrix} \frac{2\xi - 1}{2} \\ \frac{2\xi + 1}{2} \\ -2\xi \end{bmatrix}$$

3) Show that the element stiffness matrix is given by

$$\boldsymbol{K^e = \int_0^l E\,A\,B^T\,B\,dx = \int_{-1}^{1} E\,A\,B^T\,B\,J\,d\xi}$$

The rod element must be mapped using natural coordinate (ξ) to have a uniform element. Therefore, the natural coordinate must be replaced using the Jacobian by the mapped ones.

$$\mathrm{K^e(x)} = \int_{x_1}^{x_2} \mathrm{E\,A\,B^T\,B\,dx} = \int_0^L \mathrm{E\,A\,B^T\,B\,dx} \;\rightarrow\; \mathrm{K^e}(\xi) = \int_{\xi_1}^{\xi_2} \mathrm{E\,A\,B(\xi)^T\,B(\xi)\,J(\xi)\,d\xi}$$

The two extreme ends of the rod are $\xi_1 = -1$ and $\xi_2 = 1$ and the transformation of coordinate system is determined by the Jacobian.

$$\mathrm{J} = \frac{\mathrm{dx}}{\mathrm{d\xi}} \rightarrow \mathrm{dx} = \mathrm{J\,d\xi}$$

$$\mathrm{K^e}(\xi) = \int_{-1}^{1} \mathrm{E\,A\,B(\xi)^T\,B(\xi)\,J(\xi)\,d\xi}$$

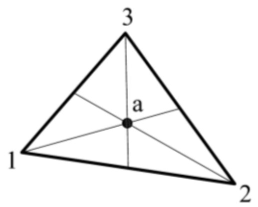
Assignment 4.2: The 3-node Plane Stress Triangle

1) Compute the entries of K^e:

The stiffness matrix can be obtained using Gauss quadrature:

$$K = \sum_{k}^{p} w_k \, B^T E \, B \, r \, J$$

- Weight (w_k): The weights for the Gauss quadrature in triangular elements for linear element are written in following figure.



n	Accuracy	Points	L_1	L_2	L_3	W_i
1	Lineal	a	1/3	1/3	1/3	1/2

- The constitutive matrix (E):

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

- Variable (r): The variable r and z are defined as,

$$r = N_1 r_1 + N_2 r_2 + N_3 r_3$$
$$z = N_1 z_1 + N_2 z_2 + N_3 z_3$$

The shape functions for linear element are:

$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$

- Jacobian (J): The half of the determinant of the Jacobian is the area.

$$J = \begin{bmatrix} \frac{\delta r}{\delta \xi} & \frac{\delta r}{\delta \eta} \\ \frac{\delta z}{\delta \xi} & \frac{\delta z}{\delta \eta} \end{bmatrix} = \begin{bmatrix} r_2 - r_1 & r_3 - r_1 \\ z_2 - z_1 & z_3 - z_1 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & b \end{bmatrix} = a\,b = 2A_{tri}$$

Every element of the Jacobian matrix is defined in the following matrix for the variable r. They are calculated in the same way for the variable z.

$$\begin{bmatrix} \frac{\delta r}{\delta \xi} \\ \frac{\delta r}{\delta \eta} \end{bmatrix} = \begin{bmatrix} \frac{\delta N_1}{\delta \xi} & \frac{\delta N_2}{\delta \xi} & \frac{\delta N_3}{\delta \xi} \\ \frac{\delta N_1}{\delta \eta} & \frac{\delta N_2}{\delta \eta} & \frac{\delta N_3}{\delta \eta} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_2 - r_1 \\ r_3 - r_1 \end{bmatrix}$$

- The strain matrix (B):

$$B=\begin{bmatrix}\frac{\delta N_1}{\delta r} & 0 & \frac{\delta N_2}{\delta r} & 0 & \frac{\delta N_3}{\delta r} & 0\\ 0 & \frac{\delta N_1}{\delta z} & 0 & \frac{\delta N_2}{\delta z} & 0 & \frac{\delta N_3}{\delta z}\\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0\\ \frac{\delta N_1}{\delta z} & \frac{\delta N_1}{\delta r} & \frac{\delta N_2}{\delta z} & \frac{\delta N_2}{\delta r} & \frac{\delta N_3}{\delta z} & \frac{\delta N_3}{\delta r}\end{bmatrix}=\begin{bmatrix}-J^{-1} & 0 & J^{-1} & 0 & 0 & 0\\ 0 & -J^{-1} & 0 & 0 & 0 & J^{-1}\\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0\\ -J^{-1} & -J^{-1} & 0 & J^{-1} & J^{-1} & 0\end{bmatrix}$$

The element of the strain matrix are defined as,

$$\begin{bmatrix}\frac{\delta N_i}{\delta r}\\ \frac{\delta N_i}{\delta z}\end{bmatrix}=J^{-1}\begin{bmatrix}\frac{\delta N_i}{\delta \xi}\\ \frac{\delta N_i}{\delta \eta}\end{bmatrix}$$

Replacing the values in the original expression,

$$K=\sum_{1}^{1} \mathrm{w_k}\,\mathrm{B^T E\, B\, r\, J}=$$

$$=\frac{1}{2}\begin{bmatrix}-J^{-1} & 0 & J^{-1} & 0 & 0 & 0\\ 0 & -J^{-1} & 0 & 0 & 0 & J^{-1}\\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0\\ -J^{-1} & -J^{-1} & 0 & J^{-1} & J^{-1} & 0\end{bmatrix}^T\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}-J^{-1} & 0 & J^{-1} & 0 & 0 & 0\\ 0 & -J^{-1} & 0 & 0 & 0 & J^{-1}\\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0\\ -J^{-1} & -J^{-1} & 0 & J^{-1} & J^{-1} & 0\end{bmatrix} r\, J$$

$$=\frac{1}{2}rJ\begin{bmatrix}-J^{-1} & 0 & \frac{N_1}{r} & -J^{-1}\\ 0 & -J^{-1} & 0 & -J^{-1}\\ J^{-1} & 0 & \frac{N_2}{r} & 0\\ 0 & 0 & 0 & J^{-1}\\ 0 & 0 & \frac{N_3}{r} & J^{-1}\\ 0 & J^{-1} & 0 & 0\end{bmatrix}\begin{bmatrix}-J^{-1} & 0 & J^{-1} & 0 & 0 & 0\\ 0 & -J^{-1} & 0 & 0 & 0 & J^{-1}\\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0\\ \frac{-1}{2J} & \frac{-1}{2J} & 0 & \frac{1}{2J} & \frac{1}{2J} & 0\end{bmatrix}=$$

$$=\begin{bmatrix}J^{-2}+\frac{N_1^2}{r^2}+\frac{1}{2J^2} & \frac{1}{2J^2} & -J^{-2}+\frac{N_1N_2}{r^2} & -\frac{1}{2J^2} & \frac{N_1N_3}{r^2}-\frac{1}{2J^2} & 0\\ \frac{1}{2J^2} & J^{-2}+\frac{1}{2J^2} & 0 & -\frac{1}{2J^2} & -\frac{1}{2J^2} & -J^{-2}\\ -J^{-2}+\frac{N_1N_2}{r^2} & 0 & J^{-2}+\frac{N_2^2}{r^2} & 0 & \frac{N_2N_3}{r^2} & 0\\ \frac{-1}{2J^2} & \frac{-1}{2J^2} & 0 & \frac{1}{2J^2} & \frac{1}{2J^2} & 0\\ \frac{N_1N_3}{r^2}-\frac{1}{2J^2} & \frac{-1}{2J^2} & \frac{N_2N_3}{r^2} & \frac{1}{2J^2} & \frac{N_3^2}{r^2}+\frac{1}{2J^2} & 0\\ 0 & -J^{-2} & 0 & 0 & 0 & J^{-2}\end{bmatrix}$$

2) Show that the sum of the rows (and columns) 2, 4 and 6 of Ke must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

The stiffness matrix will represent a force equilibrium, therefore it would be a singular matrix. But in this case due to the circumferential component is a nonsingular matrix.

$$K^e = \begin{bmatrix} J^{-2} + \frac{N_1^2}{r^2} + \frac{1}{2J^2} & \frac{1}{2J^2} & -J^{-2} + \frac{N_1 N_2}{r^2} & -\frac{1}{2J^2} & \frac{N_1 N_3}{r^2} - \frac{1}{2J^2} & 0 \\ \frac{1}{2J^2} & J^{-2} + \frac{1}{2J^2} & 0 & -\frac{1}{2J^2} & -\frac{1}{2J^2} & -J^{-2} \\ -J^{-2} + \frac{N_1 N_2}{r^2} & 0 & J^{-2} + \frac{N_2^2}{r^2} & 0 & \frac{N_2 N_3}{r^2} & 0 \\ \frac{-1}{2J^2} & \frac{-1}{2J^2} & 0 & \frac{1}{2J^2} & \frac{1}{2J^2} & 0 \\ \frac{N_1 N_3}{r^2} - \frac{1}{2J^2} & \frac{-1}{2J^2} & \frac{N_2 N_3}{r^2} & \frac{1}{2J^2} & \frac{N_3^2}{r^2} + \frac{1}{2J^2} & 0 \\ 0 & -J^{-2} & 0 & 0 & 0 & J^{-2} \end{bmatrix}$$

≠ 0	= 0	≠ 0	= 0	≠ 0	= 0

3) Compute the consistent force vector f_e for gravity forces $b = [0, -g]^T$.

The force vector due to body forces are,

$$f_{ext} = \sum_{k}^{p} w_k N^T b \, r \, J = w_k \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} b_r \\ b_z \end{bmatrix} r \, J = w_k J \left(r_1 N_1 + r_2 N_2 + r_3 N_3 \right) \begin{bmatrix} N_1 b_r \\ N_1 b_z \\ N_2 b_r \\ N_2 b_z \\ N_3 b_r \\ N_3 b_z \end{bmatrix}$$

Substituting the values of the problem,

$$f_{ext} = \frac{J}{2} \left(0 \frac{1}{3} + a \frac{1}{3} + a \frac{1}{3} \right) \begin{bmatrix} 0 \\ \frac{-g}{3} \\ 0 \\ \frac{-g}{3} \\ 0 \\ \frac{-g}{3} \end{bmatrix} = \frac{a^{2b}}{9} \begin{bmatrix} 0 \\ -g \\ 0 \\ -g \\ 0 \\ -g \end{bmatrix}$$