Computational Structural Mechanics and Dynamics **February 11, 2019**
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Assignment I:

(a) The member stiffness equations in global coordinates can be written as:

$$f^e = K^e u^e \tag{1}$$

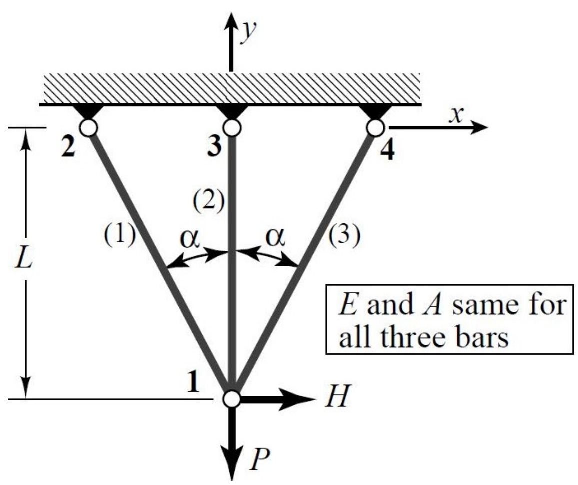
And taking into account that:

$$\bar{u}^e = T^e u^e \quad and \quad f^e = (T^e)^T \bar{f}^e \tag{2}$$

Inserting these expressions into $\bar{f}^e = \bar{K}^e \bar{u}^e$ and comparing with the member stiffness equations in global coordinates we find that the member stiffness in the global system (x, y) can be computed from the member stiffness $\bar{K}^e$ in the local system $(\bar{x}, \bar{y})$ through the congruent transformation

$$\boxed{K^e = (T^e)^T \bar{K}^e T^e} \tag{3}$$

In this case we have the following problem:



And if we define $c = cos\alpha$ and $s = sin\alpha$ we have,

$$K^1 = \frac{EAc}{L}\begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} \tag{4}$$

$$K^2 = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \tag{5}$$

$$K^3 = \frac{EAc}{L}\begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix} \tag{6}$$

Upon assembly the matrices (4), (5) and (6), the master stiffness matrix is:

$$Ku = \frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & sc^2 & 0 & 0 & -cs^2 & -sc^2 \\ & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & symm & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \tag{7}$$

The node 3 is not connected horizontally, it is only connected vertically to node 1 thus the 5th column must be zero because it represents the effects of the x-coordinate displacement on node 3, as well the 5th row that represents the effects of the displacements in each node on the internal x-coordinate forces of node 3.

(b) Apply the BCs and show the 2-equation modified stiffness system:

The system is fixed on nodes 2,3 and 4 so:

$$u_{x2} = 0; u_{y2} = 0; u_{x3} = 0; u_{y3} = 0; u_{x4} = 0; u_{y4} = 0 \tag{8}$$

We can reduce the system by eliminating the rows of the displacements that we already know and the columns that would be multiply by zero.
Then,

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{9}$$

(c) Solve for the displacements u_{x1} and u_{y1}. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \frac{\pi}{2}$. Why does u_{x1} blow up if $H \neq 0$ and $\alpha \to 0$?

Solvin the system we have that:

$$u_{x1} = \frac{HL}{2cs^2EA} \tag{10}$$

$$u_{y1} = -\frac{PL}{(1+2c^3)EA} \tag{11}$$

We have to remember that $c = cos\alpha$ and $s = sin\alpha$.

Then, if $\alpha \to 0$ it means that $c \to 1$, all bars will be aligned and the structure will offer maximum resistance to deformation, this is consistent since (11) reaches its minimum value. And if we have $\alpha \to \frac{\pi}{2}$, $c \to 0$ and (11) reaches its maximum value.

The equation for the displacement on the x direction (10) has issues with the limit cases. When we calculate displacement on the x direction for $\alpha \to \frac{\pi}{2}$ (it would mean that bars (1) and (3) are horizontal) and for $\alpha \to 0$ (all bars would be aligned and vertical) the solution blows up if $H \neq 0$.

(d) Recover the axial forces in the three members.

The axial force is given by:

$$F^e = \frac{E^eA^e}{L^e}d^e \tag{12}$$

where d^e is the elongation: $d^e = \bar{u}^e_{jx} - \bar{u}^e_{ix}$ Taking into acount that $\bar{u}^e = T^e u^e$ for each element we have:

$$\bar{u}^1 = \begin{bmatrix} \bar{u}^1_{x1} \\ \bar{u}^1_{y1} \\ \bar{u}^1_{x2} \\ \bar{u}^1_{y2} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x1} \\ u_{y1} \end{bmatrix} \tag{13}$$

$$\bar{u}^2 = \begin{bmatrix} \bar{u}^2_{x1} \\ \bar{u}^2_{y1} \\ \bar{u}^2_{x2} \\ \bar{u}^2_{y2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

$$\bar{u}^3 = \begin{bmatrix} \bar{u}^3_{x1} \\ \bar{u}^3_{y1} \\ \bar{u}^3_{x2} \\ \bar{u}^3_{y2} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

The elongation will be:

$$d^1 = (su_{x1} - cu_{y1}) - 0 = \frac{HL}{EA2cs} + \frac{PLc}{(1+2c^3)EA} \tag{16}$$

$$d^2 = 0 - (-u_{y1}) = \frac{PL}{(1+2c^3)EA} \tag{17}$$

$$d^3 = 0 - (su_{x1} + cu_{y1}) = -\frac{HL}{EA2cs} + \frac{PLc}{(1+2c^3)EA} \tag{18}$$

Finally the axial forces will be calculated by (12):

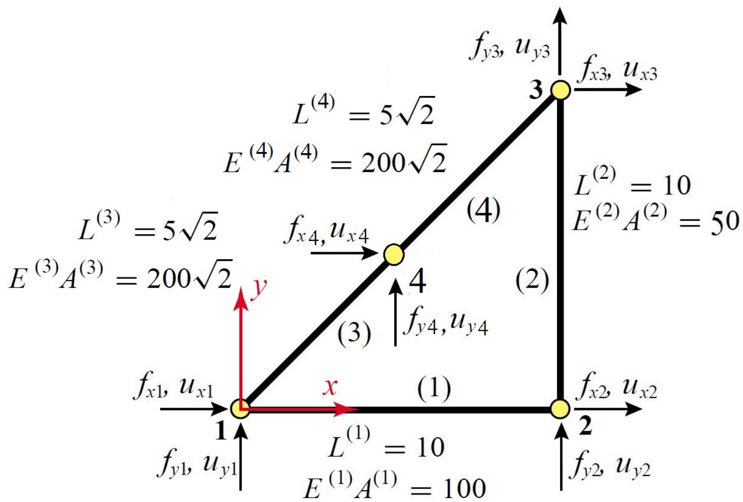
$$F^1 = \frac{H}{2s} + \frac{Pc^2}{1+2c^3} \tag{19}$$

$$F^2 = \frac{P}{1+2c^3} \tag{20}$$

$$F^3 = -\frac{H}{2s} + \frac{Pc^2}{1+2c^3} \tag{21}$$

The solution "blows up" when alpha tends to 0 and H is different from 0 for bars 1 and 3. This is due to the fact that the system is not in equilibrium for this circumstance. There is no axial force that can compensate H different from 0.

Assignment II:



The stiffnes matrix for each element will be:

$$K^1 = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

$$K^2 = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{23}$$

$$K^3 = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \tag{24}$$

$$K^4 = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \tag{25}$$

Upon assembly the matrices (22), (23), (24) and (25), the master stiffness matrix is:

$$
\begin{bmatrix}
30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\
 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\
 & & 10 & 0 & 0 & 0 & 0 & 0 \\
 & & & 5 & 0 & -5 & 0 & 0 \\
 & & & & 20 & 20 & -20 & -20 \\
 & & & & & 25 & -20 & -20 \\
 & symm & & & & & 40 & 40 \\
 & & & & & & & 40
\end{bmatrix}
\begin{bmatrix}
u_{x1}=0 \\ u_{y1}=0 \\ u_{x2} \\ u_{y2}=0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4}
\end{bmatrix}
=
\begin{bmatrix}
f_{x1} \\ f_{y1} \\ f_{x2}=0 \\ f_{y2} \\ f_{x3}=2 \\ f_{y3}=1 \\ f_{x4}=0 \\ f_{y4}=0
\end{bmatrix}
\tag{26}
$$

We can reduce the system by eliminating the rows of the displacements that we already know and the columns that would be multiply by zero:

$$
\begin{bmatrix}
10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 20 & 20 & -20 & -20 \\
0 & -5 & 20 & 25 & -20 & -20 \\
0 & 0 & -20 & -20 & 40 & 40 \\
0 & 0 & -20 & -20 & 40 & 40
\end{bmatrix}
\begin{bmatrix}
u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 2 \\ 1 \\ 0 \\ 0
\end{bmatrix}
\tag{27}
$$

As we can see the last two rows and columns are equals, the matrix is singular and the system can not be solved.

In this particular case we have that:

1. The bar properties are constant along the length
2. The only applied loads are point forces at the nodes.

Because of the foregoing conditions, we have a linear axial displacement u(x) and adding extra elements and nodes would not change the solution. For this reason this truss discretization is enough and if we add one more node the solution wont change.