Master's Degree Numerical Methods in Engineering

Computational Structural Mechanics and Dynamics

Assignment 1: Direct Stiffness Method

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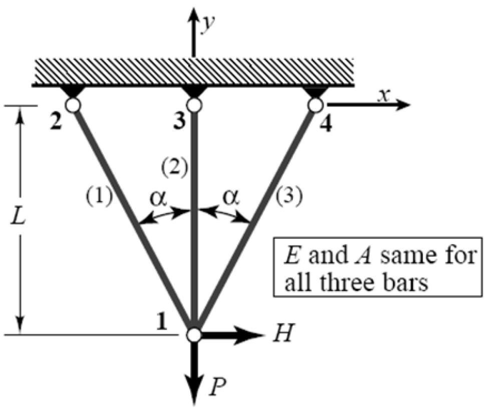
1 Problem 1

On " The Direct Stiffness Method":

Consider the truss problem defined in the Figure. All geometric and material properties: L, a, E and A, as well as the applied forces P and H, are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed-displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$.



(a) Figure 1.1. Truss structure and geometric data

2 Answer 1

(a) Show that the master stiffness equations are:

$$\frac{EA}{L}\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

in which $c = cos\ \alpha$ and $s = sin\ \alpha$. Explain from physics why the 5^{Th} row and column contain only zeros.

On the first step we discretize the model in 3 elements shown. On each node we have local coordinates from node 1 to the direction of the other nodes in the x direction and the y direction perpendicular in the right hand rule direction. Considering the global coordinate system of our problem we can see that the elements should be transferred to the global coordinate, element (1) with a *90 +* α , element (2) *90* and element (3) *90* - α transformation. Because of the 8 degrees of freedom each node has 2 displacements, one in the x and one in the y direction and two reaction forces in those directions.
For finding the global stiffness matrix we first have to develop the global stiffness matrices for each element and to assemble them after.
For elements (1) (3) we instead of using the degrees *90 +* α and *90 -* α we use the *cos 90 +* α *= sin* α *and sin 90 +* α *= - cos* α *and cos 90 -* α *= sin* α *and sin 90 -* α *= cos* α and because $L_{(1)} and L_{(3)} are L \ cos \ \alpha a cos \ \alpha will be multiplied in the stiffness matrices.$

$$K_1 = EA/L \begin{pmatrix} cs^2 & -c^2s & -cs^2 & c^2s \\ -c^2s & cs^2 & c^2s & -cs^2 \\ -cs^2 & c^2s & cs^2 & -c^2s \\ c^2s & -cs^2 & -c^2s & cs^2 \end{pmatrix}$$

$$K_2 = EA/L \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$K_3 = EA/L \begin{pmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ c^2s & cs^2 & -c^2s & -cs^2 \\ -cs^2 & -c^2s & cs^2 & c^2s \\ -c^2s & -cs^2 & c^2s & cs^2 \end{pmatrix}$$

Assembling the three matrices we reach the main stiffness matrix given by the problem.

$$\begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ \text{symm} & & & & & & & c^3 \end{bmatrix}$$

The reason that the 5^{Th} row and column of the main stiffness matrix are all zeros is that this row and column represent the x direction character of the third node and because the second element is a vertical truss so it does not have any numbers indicating its movements in the x direction. In case node number 1 was not a joint and it contained only to element 2 then the first row and column of the stiffness matrix would have been zero to.

(b) Apply the BCs and show the 2-equation modified stiffness system. Considering the fixed

condition of nodes 2,3 and 4 so $x_2 = y_2 = x_3 = y_3 = x_4 = y_4 = 0$ and at node 1 the external forces at node 1 are $F_{x1} = H$ and $F_{y1} = -P$. So canceling the rows and columns with the zero displacements the main equation will reduce into:

$$\begin{pmatrix} H \\ -P \end{pmatrix} = EA/L \begin{pmatrix} 2cs^2 & 0 \\ 0 & 2c^3+1 \end{pmatrix} \begin{pmatrix} u_{x1} \\ u_{y1} \end{pmatrix}$$

(c) Solve for the displacements $u_{x1}u_{y1}$. Check that the solution makes physical sense for the limit cases $\alpha \to 0$ and $\alpha \to \pi/2$. Why does u_{x1} "blow up" if $H \neq 0$ and $\alpha \to 0$.
Solving the above equations we will have:

$$u_{x1} = H/2cs^2$$

$$u_{y1} = -P/2c^3 + 1$$

The answers found have physical meaning for the asked limit cases. For the case of $\alpha \to 0$ all the trusses in the structure will be vertical so the structure will not be able to resist any forces applied in x direction and it will fail. For the case of $\alpha \to \pi/2$ because the length of the trusses depend on $L\ cos\ \alpha$ so in this case the $cos\alpha \to 0$ so the EA/L of the trusses will be zero.
For the case of why the answer "blows up" if $H \neq 0$ and $\alpha \to 0$, from the above solution we can see that u_{x1} depends inversely to the $cos\alpha$ and $sin^2\alpha$ so in case $\alpha \to 0$ the answer is divided by very small number and that is why it blows up.

(d) Recover the axial forces in the three members. Partial answer:
$F^{(3)} = -H/(2s) + Pc^2/(1+2c^3)$. why do $F^{(1)}$ and $F^{(3)}$ "blow up" if $H \neq 0$ and $\alpha \to 0$?

Solving the system for the reaction forces we will have:

$$F_{x2} = -H/2 - Pc^2 s/(2c^3+1)$$

$$F_{y2} = Hc/2s + Pc^3/(2c^3+1)$$

$$F_{y3} = P/(2c^3+1)$$

$$F_{x3} = 0$$

$$F_{x4} = -H/2 + Pc^2 s/(2c^3+1)$$

$$F_{y4} = -Hc/2s + Pc^3/(2c^3+1)$$

Computing the forces in the axial direction we will have:

$$F^{(1)} = H/2s + Pc^2/(2c^3+1)$$

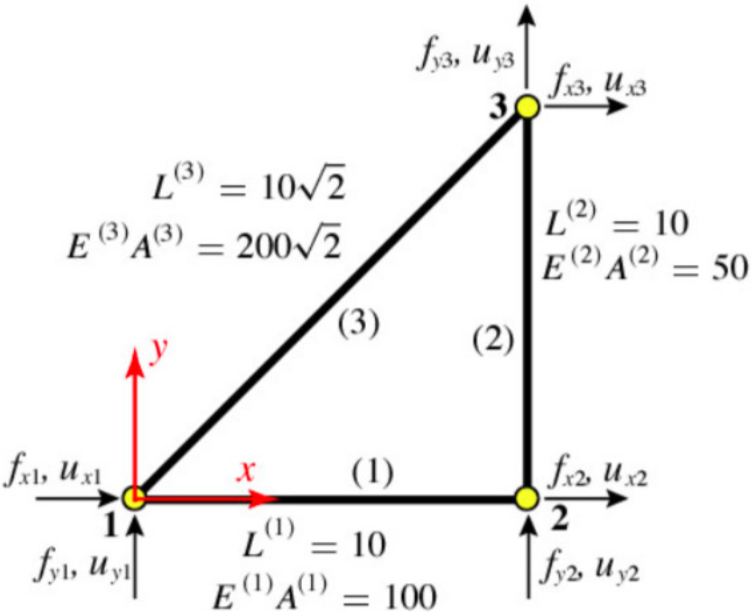
$$F^{(2)} = P/(2c^3+1)$$

$$F^{(3)} = -H/2s + Pc^2/(2c^3+1)$$

We can see that because the first term of $F^{(1)}$ and $F^{(3)}$ both depend inversely with the $sin\alpha$ then if $\alpha \to 0$ and $H \neq 0$ then the answer will blow up because it is being divided by a very small number.

3 Problem 2

Dr. Who proposes "improving" the result for the example truss of the 1^{st} lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr.Whos suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.



4 Answer 2

For this case we have to build the stiffness matrices for each of the elements. The stiffness matrices of elements (1) and (2) will be the same as before and the matrices of elements (3) and (4) will be the same because both elements have the same size and geometric characteristics.

$$K_1 = 5\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_2 = 5\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$K_3 = k_4 = 40 \begin{pmatrix} 4 & 4 & -4 & -4 \\ 4 & 4 & -4 & -4 \\ -4 & -4 & 4 & 4 \\ -4 & -4 & 4 & 4 \end{pmatrix}$$

Assembling the matrices for the global stiffness matrix and writing the equation with considering the boundary conditions of $u_{x1} = u_{y1} = u_{y2} = 0$ and $F_{x3} = 2$, $F_{y3} = 1$, $F_{x4} = 0$ and $F_{y4} = 0$ we will have:

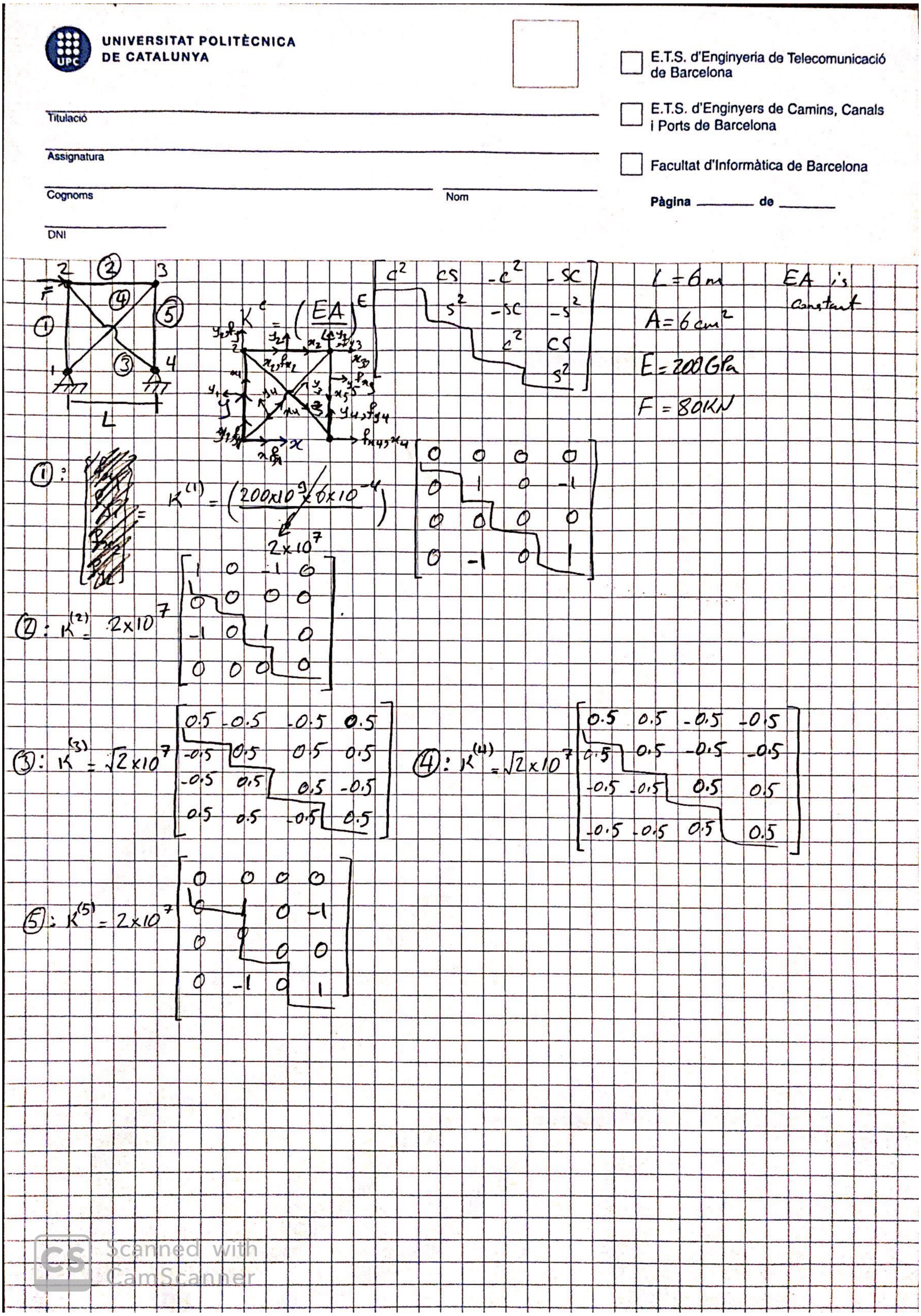
$$\begin{pmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 7 & 4 & -2 & 0 & 0 & 0 & -4 & -4 \\ 4 & 4 & 0 & 0 & 0 & 0 & -4 & -4 \\ -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 & -4 & -4 \\ 0 & 0 & 0 & -1 & 4 & 5 & -4 & -4 \\ -4 & -4 & 0 & 0 & -4 & -4 & 8 & 8 \\ -4 & -4 & 0 & 0 & -4 & -4 & 8 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_{x2} \\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{pmatrix}$$

When we reduce the system and cancel the rows and columns associated with the zero displacements, the stiffness matrix reduces to:

$$\begin{pmatrix} F_{x2} \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} 5 \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & -4 & -4 \\ 0 & 4 & 5 & -4 & -4 \\ 0 & -4 & -4 & 8 & 8 \\ 0 & -4 & -4 & 8 & 8 \end{pmatrix} \begin{pmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{pmatrix}$$

We can see that even in the reduced equation system because there are two rows exactly the same so they are linearly dependent and the stiffness matrix is singular so the solution blows up. In physical terms adding an element to the system in this case does not help to the solution and it adds unknowns to the problem without giving any new information to the problem thus it makes the problem unsolvable.

5 Classwork


UNIVERSITAT POLITÈCNICA DE CATALUNYA
Titulació
Assignatura
Cognoms
Nom
DNI
E.T.S. d'Enginyeria de Telecomunicació de Barcelona
E.T.S. d'Enginyers de Camins, Canals i Ports de Barcelona
Facultat d'Informàtica de Barcelona
Pàgina ____ de ____
L
F
x
y
$L = 6m$
EA is constant
$A = 6 cm^2$
$E = 200 GPa$
$F = 80 KN$
$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & cs & -c^2 & -sc \\ & s^2 & -sc & -s^2 \\ & & c^2 & cs \\ & & & s^2 \end{bmatrix}$
(1): $K^{(1)} = \left(\frac{200\times10^9 \times 6\times10^{-4}}{6}\right) = 2\times10^7 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$
(2): $K^{(2)} = 2\times10^7 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
(3): $K^{(3)} = \sqrt{2}\times10^7 \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix}$
(4): $K^{(4)} = \sqrt{2}\times10^7 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$
(5): $K^{(5)} = 2\times10^7 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$
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Assembly:

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \sqrt{2}\times10^{7} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 & 0 & 0 \\ 0.5 & \sqrt{2}+0.5 & 0 & -\sqrt{2} & -0.5 & -0.5 & 0 & 0 \\ 0 & 0 & \sqrt{2}+0.5 & -0.5 & -\sqrt{2} & 0 & -0.5 & 0.5 \\ 0 & -\sqrt{2} & -0.5 & \sqrt{2}+0.5 & 0 & 0 & 0.5 & 0.5 \\ -0.5 & -0.5 & -\sqrt{2} & 0 & \sqrt{2}+0.5 & 0.5 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 & 0.5 & 0.5+\sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 & -\sqrt{2} & -0.5 & 0.5+\sqrt{2} \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

Boundary conditions:

$x_1 = y_1 = x_4 = y_4 = 0$

$f_{x_2} = F = 80\,kN \quad , f_{3x} = f_{3y} = 0 = f_{2x}$

$$\begin{Bmatrix} 80\times10^{3} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \sqrt{2}\times10^{7} \begin{bmatrix} \sqrt{2}+0.5 & -0.5 & -\sqrt{2} & 0 \\ -0.5 & \sqrt{2}+0.5 & 0 & 0 \\ -\sqrt{2} & 0 & \sqrt{2}+0.5 & 0.5 \\ 0 & 0 & 0.5 & \sqrt{2}+0.5 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ x_3 \\ y_3 \end{Bmatrix} \rightarrow \begin{cases} x_2 = 8.54\times10^{-3}\,m \\ y_2 = 2.23\times10^{-3}\,m \\ x_3 = 6.77\times10^{-3}\,m \\ y_3 = -1.76\times10^{-3}\,m \end{cases}$$