Master Of Science in Computational Mechanics
Computational Structural Mechanics and Dynamics

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11 February 2019

# 1 Direct Stiffness Matrix : Assignment 1

Consider the truss problem defined in the figure 1. All geometric and material properties: L, $\alpha$, E and A, as well as the applied forces P and H are to be kept as variables. This truss has 8 degrees of freedom, with six of them removable by the fixed displacement conditions at nodes 2, 3 and 4. This structure is statically indeterminate as long as $\alpha \neq 0$. Stiffness Equation

$$f = Ku \tag{1}$$

where, f = nodal forces, K= stiffness matrix, u = nodal displacement

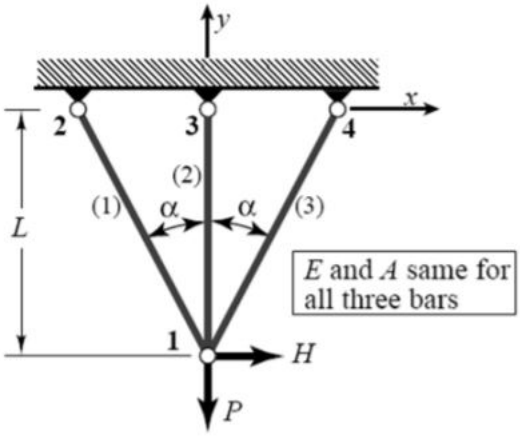


Figure 1: Truss structure. Geometry and mechanical features

## (a) master stiffness matrix

Element stiffness matrix for element 1

$$k^1 = \frac{E^1A^1}{L^1}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & sc & c^2 & sc \\ -sc & s^2 & sc & s^2 \end{bmatrix} \tag{2}$$

where, $c = cos(270 + \alpha)$ and $s = sin(270 + \alpha)$
but $cos(270 + \alpha) = sin(\alpha)$ and $sin(270 + \alpha) = -cos(\alpha)$,
$E^1 = E, A^1 = A, L^1 = L/cos(\alpha)$
therefore equation 1 becomes,

$$k^1 = \frac{EAc}{L}\begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} \tag{3}$$

where, $c = cos(\alpha), s = sin(\alpha)$
Similarly, Element stiffness matrix for element 2

$$k^2 = \frac{EA}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \tag{4}$$

Element stiffness matrix for element 3

$$k^3 = \frac{EAc}{L} \begin{bmatrix} s^2 & sc & -s^2 & -sc \\ sc & c^2 & -sc & -c^2 \\ -s^2 & -sc & s^2 & sc \\ -sc & -c^2 & sc & c^2 \end{bmatrix} \tag{5}$$

Assemble equation (3), (4), (5) into global stiffness matrix and putting it in equation (1)

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ Symm & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_x4 \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

The 5th column in the matrix represents displacement and forces in x direction at node 3. As from figure 1 we can see node 3 is only connected to bar 2 which is vertical, also element type used is truss element which can only handle axial stresses. Thus, the horizontal component should be zero.

## (b) Applying boundary conditions

Nodes 2, 3 and 4 are fixed so displacement at that nodes are zero.

$$u_{x2} = 0, u_{y2} = 0, u_{x3} = 0, u_{y3} = 0, u_x4 = 0, u_{y4} = 0$$

applying Boundary conditions to equation (6)

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} H \\ -P \end{bmatrix} \tag{7}$$

## (c) Solving for displacement

By solving system of equations obtained from equation (7),

$$\begin{bmatrix} u_{x1} \\ \\ u_{y1} \end{bmatrix} = \begin{bmatrix} \frac{HL}{2EAcs^2} \\ \\ -\frac{PL}{EA(1+2c^3)} \end{bmatrix} \tag{8}$$

Case 1: if $\alpha \to 0, c{\to}1$ and s→0. by considering equation (8) $u_{x1}$ will go to infinity acting like a pendulum causing whole equation to 'blow up' thus H should be equal to zero in this case.

Case 2: if $\alpha \to \pi/2$, c→ 1 and s→ 0 the bar 1 and 3 will tend to go horizontal going near to wall and will deform minimum.

## (d) Axial Forces in members

Axial force is determined by,

$$F^e = \frac{E^e A^e}{L^e} d^e \tag{9}$$

where $d^e$ is elongation, $d^e = \bar{u}^e_{jx} - \bar{u}^e_{ix}$ and $\bar{U}^e = T^e u^e$

$$\bar{u}^1 = \begin{bmatrix} \bar{u}^1_{x1} \\ \bar{u}^1_{y1} \\ \bar{u}^1_{x2} \\ \bar{u}^1_{y2} \end{bmatrix} = \begin{bmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x1} \\ u_{y1} \end{bmatrix} \tag{10}$$

$$\bar{u}^2 = \begin{bmatrix} \bar{u}^2_{x1} \\ \bar{u}^2_{y1} \\ \bar{u}^2_{x2} \\ \bar{u}^2_{y2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\bar{u}^3 = \begin{bmatrix} \bar{u}^3_{x1} \\ \bar{u}^3_{y1} \\ \bar{u}^3_{x2} \\ \bar{u}^3_{y2} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

$$d^1 = (su_{x1} - cu_{y1}) - 0 = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)} \tag{13}$$

$$d^2 = 0 - (-u_{y1) = \frac{PL}{EA(1+2c^3)}} \tag{14}$$

$$d^3 = 0 - (su_{x1} + cu_{y1}) = -\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)} \tag{15}$$

Therefore,

$$F^1 = \frac{H}{2s} + \frac{Pc^2}{1+2c^3} \tag{16}$$

$$F^2 = \frac{P}{1+2c^3} \tag{17}$$

$$F^3 = -\frac{H}{2s} + \frac{Pc^2}{1+2c^3} \tag{18}$$

As when $\alpha \to 0$ solution 'blows up' if $H \neq 0$, thus the axial forces doesn't makeup to the external forces.

# 2 Assignment 2

Dr. Who proposes "improving" the result for the example truss of the 1st lesson by putting one extra node, 4 at the midpoint of member (3) 1-3, so that it is subdivided in two different members: (3) 1-4 and (4) 3-4. His "reasoning" is that more is better. Try Dr. Who's suggestion by hand computations and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically.

consider the example in figure 2 Example,

$$K^1 = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

$$K^2 = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{20}$$

$$K^3 = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \tag{21}$$

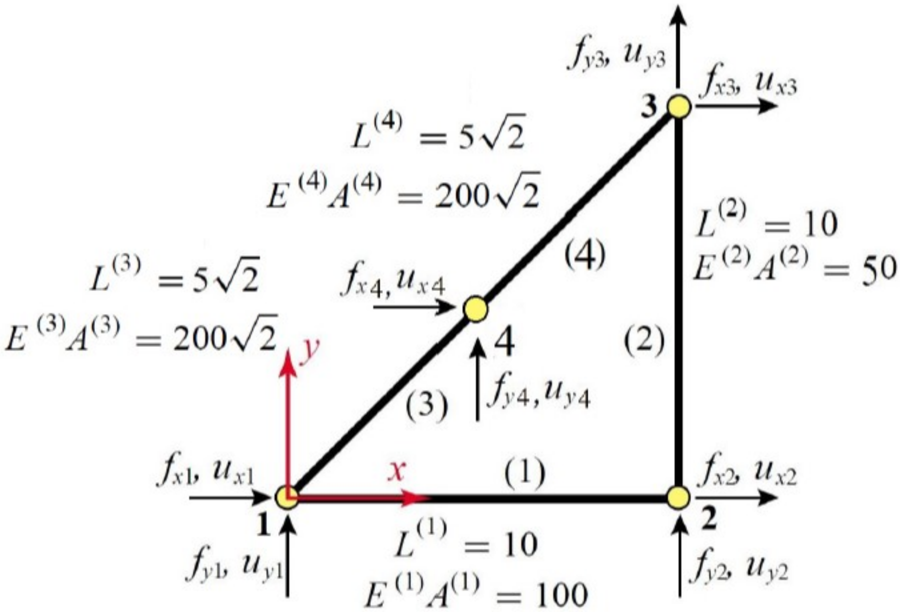


Figure 2: Truss structure. Geometry and mechanical features

$$K^4 = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \tag{22}$$

Assemblying (19),(20),(21) and (22) into global stiffness matrix $K = K^1 + k^2 + K^3 + K^4$

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix} \tag{23}$$

Boundary conditions,
$F_{x2} = 0, F_{x3} = 2, F_{y3} = 1, F_{x4} = 0, F_{y4} = 0$
$u_{x1} = 0, u_{y1} = 0, u_{y2} = 0$

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 20 & -20 & -20 \\ 0 & -5 & 20 & 25 & -20 & -220 \\ 0 & 0 & -20 & -20 & 40 & 40 \\ 0 & 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

Inference: As from the reduced equation (24) it is observed that the last 2 rows and column of the matrix representing 4th node makes the matrix singular and could not be solved moreover speaking physically without proper boundary conditions at the 4th node the system will not be in equilibrium as there may consist rotation movement also.