Computational Structural Mechanics and Dynamics

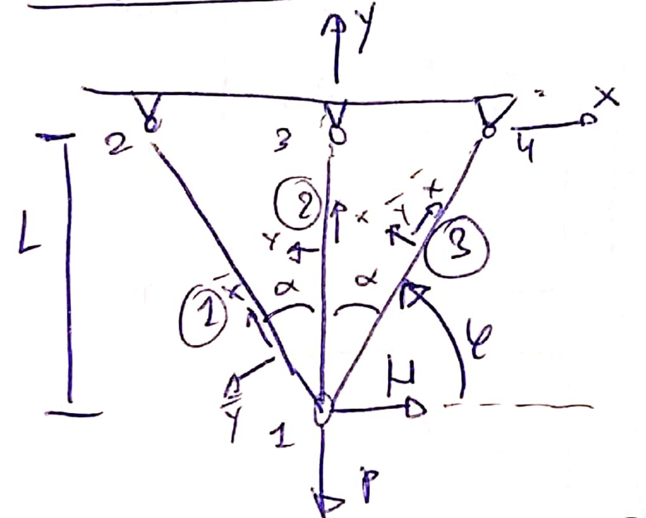
ASSIGNMENT 1:
DIRECT STIFFNESS METHOD

YEAR 2020

NAME:
IVÁN PÉREZ GARCÍA.

CONTENT
- Assignment 1
- Assignment 2
- CLASS PROBLEM.

Assigment 1



$$\boxed{L_1 = L_3}$$

$$L_1 \cos\alpha = L \rightarrow \boxed{L_1 = \frac{L}{\cos\alpha}}$$

$$\boxed{\begin{array}{l} E_1 = E_2 = E_3 \\ A_1 = A_2 = A_3 \end{array}}$$

element	n_1	n_2	φ	$\cos\varphi$ $C(\varphi)$	$\sin\varphi$ $S(\varphi)$
①	1	2	$90+\alpha$	$-\sin\alpha$	$\cos\alpha$
②	1	3	90°	0	1
③	1	4	$(90-\alpha)$	$\sin\alpha$	$\cos\alpha$

General k^e for φ:

$$\left(\frac{EA}{L}\right)^e \begin{pmatrix} c^2 & sc & -c^2 & -sc \\ & s^2 & -sc & -s^2 \\ & & c^2 & sc \\ & & & s^2 \end{pmatrix}$$

Stiffness matrices for each element in global coordinates dependent of α:

①

$$K^1 = \left(\frac{EA}{L}\right) \cdot c \begin{array}{c|cccc} & 1 & & 2 & \\ \hline 1 & s^2 & -sc & -s^2 & +sc \\ & -sc & c^2 & sc & -c^2 \\ 2 & -s^2 & sc & s^2 & -sc \\ & sc & -c^2 & -sc & c^2 \end{array} = \left(\frac{EA}{L}\right) \begin{array}{c|cccc} & 1 & & 2 & \\ \hline 1 & cs^2 & -sc^2 & -s^2c & sc^2 \\ & -sc^2 & c^3 & sc^2 & -c^3 \\ 2 & -s^2c & sc^2 & cs^2 & -sc^2 \\ & sc^2 & -c^3 & -sc^2 & c^3 \end{array}$$

②

$$K^2 = \left(\frac{EA}{L}\right)^2 \begin{array}{c|cccc} & 1 & & 3 & \\ \hline 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & -1 \\ & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 1 \end{array}$$

③

$$K^3 = \left(\frac{EA}{L}\right) \begin{array}{c|cccc} & 1 & & 4 & \\ \hline 1 & cs^2 & sc^2 & -s^2c & -sc^2 \\ & sc^2 & c^3 & -sc^2 & -c^2 \\ & -s^2c & -sc^2 & cs^2 & sc^2 \\ 4 & -sc^2 & -c^2 & sc^2 & c^3 \end{array}$$

Assembly of whole System stiffness matrix

node	elements
1	①②③
2	①
3	②
4	③

$$\begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 2cs^2 & 0 & -s^2c & sc^2 & 0 & 0 & -s^2c & -sc^2 \\ 0 & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -sc^2 & -c^2 \\ -s^2c & sc^2 & cs^2 & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -s^2c & -sc^2 & 0 & 0 & 0 & 0 & cs^2 & sc^2 \\ -sc^2 & -c^2 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{pmatrix} \cdot \begin{bmatrix} M_{x1} \\ M_{y1} \\ M_{x2} \\ M_{y2} \\ M_{x3} \\ M_{y3} \\ M_{x4} \\ M_{y4} \end{bmatrix}$$

row 5th is the force in X-direction corresponding to the node 3. Since the node 3 belongs to bar/truss ② which is in vertical position and no bending moment is considerede for the element (only axial fore), any displacement of node 1 cannot trasmit force in X global direction (Y local direction).

b) Modified stiffness system after aplying BC's

- $[M_{x2} = M_{y2} = M_{x3} = M_{y3} = M_{x4} = M_{y4} = 0]$

Reduced system:

$$\begin{bmatrix} H \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 \\ 0 & 1+2c^3 \end{bmatrix} \begin{bmatrix} M_{x1} \\ M_{y1} \end{bmatrix} \rightarrow \begin{cases} \left[M_{x1} = \dfrac{HL}{EA(2cs^2)} \right] \\ \left[M_{y1} = \dfrac{-PL}{EA(1+2c^3)} \right] \end{cases}$$

c)

α	$\sin\alpha$	$\cos\alpha$
$\alpha \to 0$	0	1
$\alpha \to \pi/2$	1	0

$\boxed{\alpha \to 0}$

$$\bullet\ \bar{\mu}_{x_1} = \lim_{\alpha \to 0} \underbrace{\frac{L}{EA}}_{k} \cdot \frac{H}{2 s^2} = \lim_{\alpha \to 0} k \cdot \frac{H}{s^2} = +\infty$$

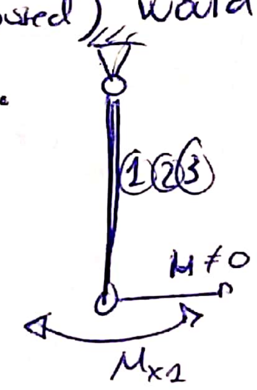
$$\bullet\ \mu_{y_1} = \lim_{\alpha \to 0} \frac{-PL}{EA(1+2c^3)} = \frac{-PL}{3EA}$$

$\boxed{\alpha \to \pi/2}$

$$\bullet\ \mu_{x_1} = \lim_{\alpha \to \pi/2} \frac{H}{2EA} \overbrace{\frac{L}{c}}^{L^e \to \infty} = +\infty$$

$$\bullet\ \mu_{y_1} = \lim_{\alpha \to \pi/2} \frac{-PL}{EA(1+2c^3)} = \frac{-PL}{EA}$$

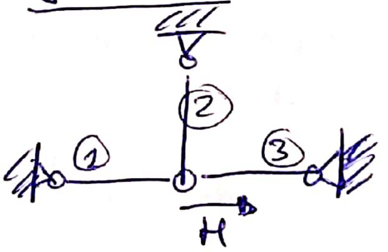
- for $\alpha \to 0$ μ_{x_1} blows up if $H \neq 0$, due to the sistem will be "UNCONSTRAINED" in x-direction and the 3 "merged" bars (same geometrical position, superposed) would move freely.



- for $\alpha \to \frac{\pi}{2}$, μ_{x_1} has only physical sense if we constrain the relation of $\left(\frac{L}{c}\right)$ wich is the length of bar 1 and 3. For instue if $L^1 = L^3 = L^2$.

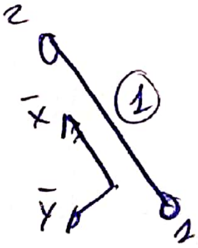
then: $$\mu_{x_1} = \lim_{\alpha \to \pi/2} \frac{HL}{\textcircled{2}EA}$$

"2" beause it's equivalent to pull/push both bars.

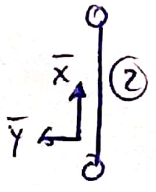


d) $$\bar{F}^e = \frac{E^eA}{L^e} d^e = \frac{E^eA^e}{L^e}\left(\bar{\mu}_{xj}^e - \bar{\mu}_{xi}^e\right)$$

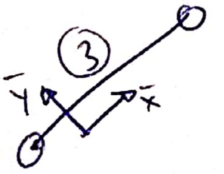
① $$\bar{\mu}_1^1 = T^1 \mu_1 = \begin{bmatrix} -s & c \\ -c & -s \end{bmatrix} \begin{bmatrix} \mu_{x1} \\ \mu_{y1} \end{bmatrix} = \begin{bmatrix} \bar{\mu}_{x1}^1 \\ \bar{\mu}_{y1}^1 \end{bmatrix}$$



② $$\bar{\mu}_1^2 = T^2 \mu_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mu_{x1} \\ \mu_{y1} \end{bmatrix} = \begin{bmatrix} \bar{\mu}_{x1}^2 \\ \bar{\mu}_{y1}^2 \end{bmatrix}$$



③ $$\bar{\mu}_1^3 = T^3 \mu_3 = \begin{bmatrix} s & c \\ -c & s \end{bmatrix} \begin{bmatrix} \mu_{x1} \\ \mu_{y1} \end{bmatrix} = \begin{bmatrix} \bar{\mu}_{x1}^3 \\ \bar{\mu}_{y1}^3 \end{bmatrix}$$



Axial forces of 3 bars :

① $F^1 = \frac{EA}{(L/c)}\left(0-\left[-SM_{x1}+CM_{y1}\right]\right) = \frac{EA}{L}\left(cSM_{x2}+c^2M_{y1}\right) =$

$$= \frac{\cancel{EA}}{\cancel{L}}\left[cs\left(\frac{H\cancel{L}}{\cancel{EA}(2cs^2)}\right)-c^2\left(\frac{-P\cancel{L}}{\cancel{EA}(1+2c^3)}\right)\right] = \boxed{\frac{H}{2s}+\frac{Pc^2}{(1+2c^3)} = F^1}$$

② $F^2 = \frac{EA}{L}\left(0-\left[M_{y1}\right]\right) = \boxed{\frac{P}{(1+2c^3)} = F^2}$

③ $F^3 = \frac{EA}{(L/c)}\left(0-\left[SM_{x2}+CM_{y1}\right]\right) = -csM_{x1}-c^2M_{y1} =$

$$= -cs\left(\frac{H}{2cs^2}\right)-c^2\left(\frac{-P}{(1+2c^3)}\right) = \boxed{\left(\frac{Pc^2}{1+2c^3}\right)-\left(\frac{H}{2s}\right) = F^3}$$

if,

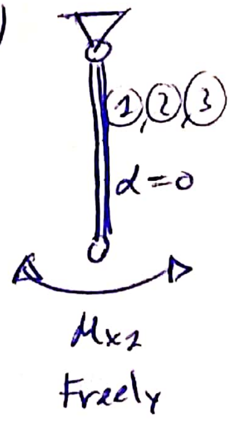
$\boxed{\alpha \to 0}$ $s=0;\ c=1$ ① $\longrightarrow F^1 \to +\infty$ if $H \neq 0$

if $H=0 \to \boxed{F^1 = \frac{P}{3}}$

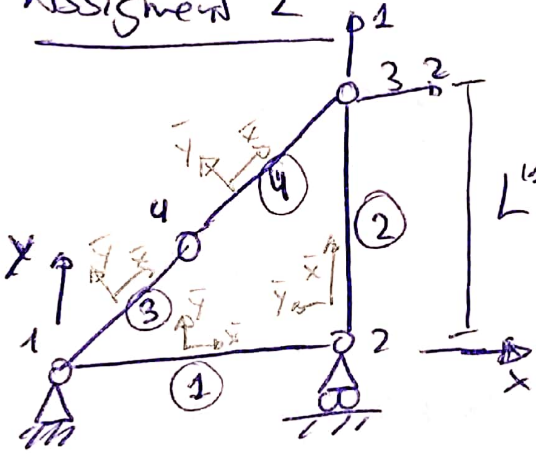
Due to system unconstrained in X-global direction

③ $\longrightarrow F^3 \to -\infty$ if $H \neq 0$

if $H=0 \to \boxed{F^3 = \frac{P}{3}}$



Assigmend 2



$L^3 = L^4 = \frac{10\sqrt{2}}{2}$

$L^{(1)} = L^{(2)} = 10$

We can take the globalized elements stiffness equations for each element from the example of class; So that:

for bar ③ and ④ $\rightarrow \frac{EA}{L_3} = \frac{200\sqrt{2}}{(10\sqrt{2}/2)} = 40$

① $$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{bmatrix} \begin{array}{c} 1 \\ \\ 2 \end{array} = \begin{array}{cc} 1 & 2 \\ \end{array} \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}$$

③ $$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x4} \\ f_{y4} \end{bmatrix} \begin{array}{c} 1 \\ \\ 4 \end{array} = \begin{array}{cc} 1 & 4 \\ \end{array} \begin{bmatrix} 20 & 20 & -20 & -20 \\ 20 & 20 & -20 & -20 \\ -20 & -20 & 20 & 20 \\ -20 & -20 & 20 & 20 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

② $$\begin{bmatrix} f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} \begin{array}{c} 2 \\ \\ 3 \end{array} = \begin{array}{cc} 2 & 3 \\ \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

④ $$\begin{bmatrix} f_{x4} \\ f_{y4} \\ f_{x3} \\ f_{y3} \end{bmatrix} \begin{array}{c} 4 \\ \\ 3 \end{array} = \begin{array}{cc} 4 & 3 \\ \end{array} \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix} \begin{bmatrix} u_{x4} \\ u_{y4} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Global stiffness system:

$$\begin{bmatrix} f_{x1} \\ f_{y2} \\ \cancel{f_{x2}}\ 0 \\ f_{y2} \\ \cancel{f_{x3}} \to 1 \\ \cancel{f_{y3}} \to 2 \\ f_{x4} \\ f_{y4} \end{bmatrix} = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ 20 & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ -10 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & 40 & 40 \\ 0 & 0 & 0 & -5 & 20 & 25 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 40 & 40 \\ -20 & -20 & 0 & 0 & -20 & -20 & 46 & 40 \end{bmatrix} \cdot \begin{bmatrix} \cancel{u_{x1}}\ 0 \\ \cancel{u_{y1}}\ 0 \\ u_{x2} \\ \cancel{u_{y2}}\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} \rightarrow f = K \cdot u$$

Boundary conditions: $\underbrace{[u_{x1} = u_{y1} = u_{y2} = 0]}_{3}$ $\underbrace{[f_{x2} = 0 \; ; \; f_{x3} = 2 \; ; \; f_{y3} = 1]}_{3}$

⑤

Reduced System:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & 40 & 40 \\ 0 & 20 & 25 & 40 & 40 \end{bmatrix} \cdot \begin{bmatrix} M_{x2} \\ M_{x3} \\ M_{y3} \\ M_{x4} \\ M_{y4} \end{bmatrix} \rightarrow M_{x2} = 0$$

System <u>underdetermined</u>

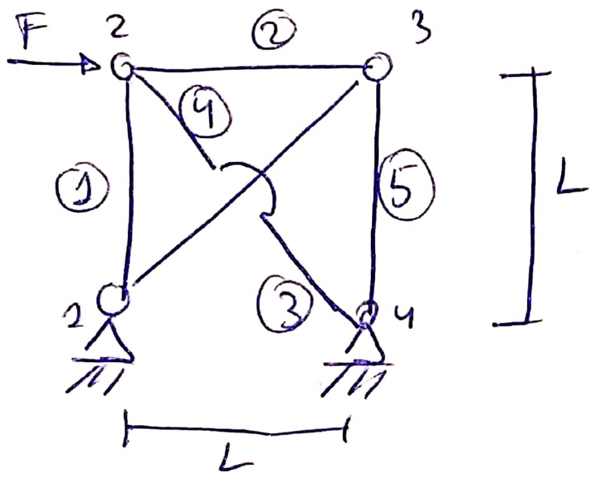
- 4 nodes → 8 DoF
- Only 6 BC's

→ <u>the system has 2 DoF.</u>

there is needed 2 more BC's to contrain the system.

For istance to fix the 2 DoF of node 4, will solve the system.

CLASS PROBLEM :



$L = 6m$

$E = 200 GPa$

$F = 80 kN$

$A = 6 cm^2$

$$\frac{EA}{L} = 200 \cdot 10^5$$

- bar 4 and 3:

$$L_3 = L_4 = \frac{L}{\left(\frac{\sqrt{2}}{2}\right)} \rightarrow$$

$$\rightarrow \left[\frac{EA}{L}\frac{\sqrt{2}}{2}\right]^4 = \left[\frac{EA}{L}\left(\frac{\sqrt{2}}{2}\right)\right]^3$$

elemental stiffness matrix in global coordenates:

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \left(\frac{EA}{L}\right) \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ & s^2 & -sc & -s^2 \\ & & c^2 & sc \\ & \text{sym.} & & s^2 \end{bmatrix} \cdot \begin{bmatrix} \mu_{xi} \\ \mu_{yi} \\ \mu_{xj} \\ \mu_{yj} \end{bmatrix}$$

element	n_i	n_j	α	c	s
①	1	2	90°	0	1
②	2	3	0°	1	0
③	2	4	-45°	$\sqrt{2}/2$	$-\sqrt{2}/2$
④	1	3	+45°	$\sqrt{2}/2$	$\sqrt{2}/2$
⑤	3	4	-90°	0	-1

elemental stiffness matrices:

① $$K^1 = \left(\frac{EA}{L}\right) \begin{array}{c|cc|cc} & 1 & & 2 & \\ \hline 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & -1 \\ \hline 2 & 0 & 0 & 0 & 0 \\ & 0 & -1 & 0 & 1 \end{array}$$

③ $$K^3 = \left(\frac{EA}{L}\right)\frac{\sqrt{2}}{2} \begin{array}{c|cc|cc} & 2 & & 4 & \\ \hline 2 & 0,5 & -0,5 & -0,5 & 0,5 \\ & -0,5 & 0,5 & 0,5 & 0,5 \\ \hline 4 & -0,5 & 0,5 & 0,5 & -0,5 \\ & 0,5 & -0,5 & -0,5 & 0,5 \end{array}$$

② $$K^2 = \frac{EA}{L} \begin{array}{c|cc|cc} & 2 & & 3 & \\ \hline 2 & 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline 3 & -1 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \end{array}$$

④ $$K^4 = \left(\frac{EA}{L}\right)\frac{\sqrt{2}}{2} \begin{array}{c|cc|cc} & 1 & & 3 & \\ \hline 1 & 0,5 & 0,5 & -0,5 & -0,5 \\ & 0,5 & 0,5 & -0,5 & -0,5 \\ \hline 3 & -0,5 & -0,5 & 0,5 & 0,5 \\ & -0,5 & -0,5 & 0,5 & 0,5 \end{array}$$

$$⑤\ K^5 = \left(\frac{EA}{L}\right) \begin{array}{c|cc|cc} & \multicolumn{2}{c|}{3} & \multicolumn{2}{c}{4} \\ 3 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & -1 \\ \hline 4 & 0 & 0 & 0 & 0 \\ & 0 & -1 & 0 & 1 \end{array}$$

$$\frac{EA}{L} = 200 \cdot 10^5$$

Global stiffness matrix & BC's :

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ F \\ 0 \\ 0 \\ 0 \\ f_{x4} \\ f_{y4} \end{bmatrix} = (200 \cdot 10^5) \left[\begin{array}{cc|cc|cc|cc} \sqrt{2}/4 & \sqrt{2}/4 & 0 & 0 & -\sqrt{2}/4 & -\sqrt{2}/4 & 0 & 0 \\ \sqrt{2}/4 & (1+\sqrt{2}/4) & 0 & -1 & -\sqrt{2}/4 & -\sqrt{2}/4 & 0 & 0 \\ \hline 0 & 0 & (1+\frac{\sqrt{2}}{4}) & -\frac{\sqrt{2}}{4} & -1 & 0 & -\sqrt{2}/4 & \sqrt{2}/4 \\ 0 & -1 & -\frac{\sqrt{2}}{4} & (1+\sqrt{2}/4) & 0 & 0 & \sqrt{2}/4 & \sqrt{2}/4 \\ \hline -\sqrt{2}/4 & -\sqrt{2}/4 & -1 & 0 & (1+\frac{\sqrt{2}}{4}) & \frac{\sqrt{2}}{4} & 0 & 0 \\ -\sqrt{2}/4 & -\sqrt{2}/4 & 0 & 0 & \frac{\sqrt{2}}{4} & (1+\frac{\sqrt{2}}{4}) & 0 & -1 \\ \hline 0 & 0 & -\sqrt{2}/4 & \sqrt{2}/4 & 0 & 0 & \sqrt{2}/4 & -\sqrt{2}/4 \\ 0 & 0 & \sqrt{2}/4 & -\sqrt{2}/4 & 0 & -1 & -\frac{\sqrt{2}}{4} & (1+\frac{\sqrt{2}}{4}) \end{array}\right] \cdot \begin{bmatrix} 0 \\ 0 \\ \mu_{x2} \\ \mu_{y2} \\ \mu_{x3} \\ \mu_{y3} \\ 0 \\ 0 \end{bmatrix}$$

Reduced System

$$\begin{bmatrix} F \\ 0 \\ 0 \\ 0 \end{bmatrix} = 200 \cdot 10^5 \left[\begin{array}{cc|cc} (1+\frac{\sqrt{2}}{4}) & -\sqrt{2}/4 & -1 & 0 \\ -\sqrt{2}/4 & (1+\sqrt{2}/4) & 0 & 0 \\ \hline -1 & 0 & (1+\sqrt{2}/4) & \sqrt{2}/4 \\ 0 & 0 & \sqrt{2}/4 & (1+\sqrt{2}/4) \end{array}\right] \cdot \begin{bmatrix} \mu_{x2} \\ \mu_{y2} \\ \mu_{x3} \\ \mu_{y3} \end{bmatrix}$$

Solution using Python code :

$$\boxed{\begin{bmatrix} \mu_{x2} \\ \mu_{y2} \\ \mu_{x3} \\ \mu_{y3} \end{bmatrix} = \begin{bmatrix} 0{,}00854 \\ 0{,}00223 \\ 0{,}00677 \\ -0{,}00177 \end{bmatrix} m}$$