Assignment 1

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Part 1

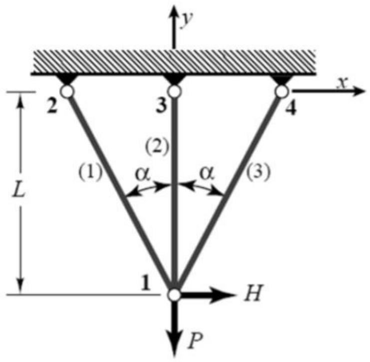


Figure 1: Three Bars Truss Problem

a) Each bar is considered as an element, numbered as shown in the figure 1. The angles with the horizontal for each element are the following:

$$\theta_1 = 90^\circ + \alpha$$
$$\theta_2 = 90^\circ$$
$$\theta_3 = 90^\circ - \alpha$$

Having $s = sin(\alpha)$, and $c = cos(\alpha)$, the following expressions will be fixed to assemble the stiffness matrix:

$$\begin{aligned} sin(\theta_1) &= c \qquad cos(\theta_1) = -s \\ sin(\theta_2) &= 1 \qquad cos(\theta_2) = 0 \\ sin(\theta_3) &= c \qquad cos(\theta_3) = s \end{aligned}$$

In this problem, there are 3 elements with a total of 4 nodes, each node with 2 degrees of freedom. Therefore the result of the master stiffness equation will be an 8×8 matrix built with 3 stiffness matrices from the 3 bars, each matrix will be 4×4 and have the following formula:

$$K^e = \left(\frac{EA}{L}\right)^e \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

Therefore the following matrices are obtained:

$$K^1 = \left(\frac{EA}{L/c}\right)^1 \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} = \left(\frac{EA}{L}\right)^1 \begin{bmatrix} cs^2 & -c^2s & -cs^2 & c^2s \\ -c^2s & c^3 & c^2s & -c^3 \\ -cs^2 & c^2s & cs^2 & -c^2s \\ c^2s & -c^3 & -c^2s & c^3 \end{bmatrix}$$

$$K^2 = \left(\frac{EA}{L}\right)^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^3 = \left(\frac{EA}{L}\right)^3 \begin{bmatrix} cs^2 & c^2s & -cs^2 & -c^2s \\ c^2s & c^3 & -c^2s & -c^3 \\ -cs^2 & -c^2s & cs^2 & c^2s \\ -c^2s & -c^3 & c^2s & c^3 \end{bmatrix}$$

Assembling these matrices in the global matrix, with $\frac{EA}{L}$ as a common factor, the resultant stiffness matrix is the following:

$$K = \frac{EA}{L} \begin{bmatrix} cs^2+cs^2 & c^2s-c^2s & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & c^3+1+c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & sym & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}$$

$$K = \frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & sym & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix}$$

Each node has two degrees of freedom (in the x and y directions), therefore the deformation vector will be the following:

$$u = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

There are forces for each node in the x and y directions. In this cases, only two forces are applied on node 1, therefore the force vector will be the following:

$$f = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting the equation together:

$$Ku = f$$

$$\frac{EA}{L} \begin{bmatrix} 2cs^2 & 0 & -cs^2 & c^2s & 0 & 0 & -cs^2 & -c^2s \\ & 1+2c^3 & c^2s & -c^3 & 0 & -1 & -c^2s & -c^3 \\ & & cs^2 & -c^2s & 0 & 0 & 0 & 0 \\ & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & sym & & & 1 & 0 & 0 \\ & & & & & & cs^2 & c^2s \\ & & & & & & & c^3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The bar element is perpendicular to the x direction if node 3, and the forces are on the same element on node 1, and following the direction matrix,u_{x3} doesn't have any effect on the displacement even if it's not fixed in the x direction, this explains

b) Nodes 2,3 and 4 don't move, therefore their displacements in both x and y will be 0.

$$u = \begin{bmatrix} u_{x1} \\ u_{y1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This leads to only two unknowns, and the other values that are multiplied by 0 can be removed from the stiffness matrix. Therefore two equations are enough to solve this problem, which are the first and second rows of the matrix:

$$\frac{EA}{L}\begin{bmatrix}2cs^2 & 0\\ 0 & 1+2c^3\end{bmatrix}\begin{bmatrix}u_{x1}\\ u_{y1}\end{bmatrix}=\begin{bmatrix}H\\ -P\end{bmatrix}$$

c) Solving the above equation for each displacement:

$$u_{x1} = \frac{HL}{2EAcs^2}$$
$$u_{y1} = \frac{-PL}{EA}(\frac{1}{1+2c^3})$$

When $\alpha \to 0$ and $\alpha \to \frac{\pi}{2}$, the following displacements are obtained respectively:

$$u_{x1} = \frac{HL}{2EA\times 0} = \infty \qquad u_{x1} = \frac{HL}{2EA\times 0} = \infty$$
$$u_{y1} = \frac{-PL}{EA}(\frac{1}{1+2}) = \frac{-PL}{3EA} \qquad u_{y1} = \frac{-PL}{EA}(\frac{1}{1+2\times 0}) = \frac{-PL}{EA}$$

From the answers above, if $H \neq 0$ and $\alpha \to 0$, the answer "blows up"

d) Convert the displacement matrix to the local displacement for each element:

$$\bar{u}^e = T^e u^e$$

$$\begin{bmatrix}\bar{u_{xi}}\\ \bar{u_{yi}}\\ \bar{u_{xj}}\\ \bar{u_{yj}}\end{bmatrix}=\begin{bmatrix}c & s & 0 & 0\\ -s & c & 0 & 0\\ 0 & 0 & c & s\\ 0 & 0 & -s & c\end{bmatrix}\begin{bmatrix}u_{xi}\\ u_{yi}\\ u_{xj}\\ u_{yj}\end{bmatrix}$$

Elements from 1 to 3 will be solved in function of α.
Element 1:

$$\begin{bmatrix}\bar{u_{x1}}\\ \bar{u_{y1}}\\ \bar{u_{x2}}\\ \bar{u_{y2}}\end{bmatrix}=\begin{bmatrix}-s & c & 0 & 0\\ -c & -s & 0 & 0\\ 0 & 0 & -s & c\\ 0 & 0 & -c & -s\end{bmatrix}\begin{bmatrix}u_{x1}\\ u_{y1}\\ u_{x2}\\ u_{y2}\end{bmatrix}$$

$$\begin{bmatrix}\bar{u_{x1}}\\ \bar{u_{y1}}\\ \bar{u_{x2}}\\ \bar{u_{y2}}\end{bmatrix}=\begin{bmatrix}-\frac{HL}{2EAcs}-\frac{PLc}{EA(1+2c^3)}\\ -\frac{HL}{2EAs^2}+\frac{PLs}{EA(1+2c^3)}\\ 0\\ 0\end{bmatrix}$$

Elongation: $d = -\bar{u_{x1}} = \frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$
Axial Force: $F^{(1)} = \frac{EA}{L/c}d = \frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}$
Element 2:

$$\begin{bmatrix} \bar{u_{x1}} \\ \bar{u_{y1}} \\ \bar{u_{x3}} \\ \bar{u_{y3}} \end{bmatrix} = \begin{bmatrix} -0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

$$\begin{bmatrix} \bar{u_{x1}} \\ \bar{u_{y1}} \\ \bar{u_{x3}} \\ \bar{u_{y3}} \end{bmatrix} = \begin{bmatrix} -\frac{PL}{EA(1+2c^3)} \\ -\frac{HL}{2EAcs^2} \\ 0 \\ 0 \end{bmatrix}$$

Elongation: $d = -\bar{u_{x1}} = \frac{PL}{EA(1+2c^3)}$
Axial Force: $F^{(2)} = \frac{EA}{L}d = \frac{P}{(1+2c^3)}$
Element 3:

$$\begin{bmatrix} \bar{u_{x1}} \\ \bar{u_{y1}} \\ \bar{u_{x4}} \\ \bar{u_{y4}} \end{bmatrix} = \begin{bmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

$$\begin{bmatrix} \bar{u_{x1}} \\ \bar{u_{y1}} \\ \bar{u_{x4}} \\ \bar{u_{y4}} \end{bmatrix} = \begin{bmatrix} \frac{HL}{2EAcs} - \frac{PLc}{EA(1+2c^3)} \\ -\frac{HL}{2EAs^2} - \frac{PLs}{EA(1+2c^3)} \\ 0 \\ 0 \end{bmatrix}$$

Elongation: $d = -\bar{u_{x1}} = -\frac{HL}{2EAcs} + \frac{PLc}{EA(1+2c^3)}$
Axial Force: $F^{(3)} = \frac{EA}{L/c}d = -\frac{H}{2s} + \frac{Pc^2}{(1+2c^3)}$
for $H \neq 0$ and $\alpha \to 0$, $\frac{H}{2s} \to \frac{H}{2 \times 0} \to \infty$

Part 2

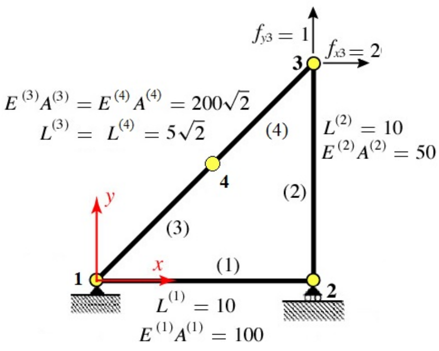


Figure 2: Four Bars Truss Problem

Each bar is considered as an element, numbered as shown in the figure 2. The angles with the horizontal for each element are the following:

$$\theta_1 = 0°$$
$$\theta_2 = 90°$$
$$\theta_{3-4} = 45°$$

In this problem, there are 4 elements with 4 nodes, each node with 2 degrees of freedom. Therefore the result of the master stiffness equation will be a 8×8 matrix built with 4 stiffness matrices from the 4 bars.

Therefore the following matrices are obtained:

$$K^1 = \left(\frac{EA}{L}\right)^1 \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^2 = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$K^3 = K^4 = 40 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

Assembling these matrices in the global matrix, the resultant stiffness matrix is the following:

$$K = \begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix}$$

Each node has two degrees of freedom (in the x and y directions), therefore the deformation vector will be the following:

$$u = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

There are forces for each node in the x and y directions. In this cases, only two forces are applied on node 1, therefore the force vector will be the following:

$$f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Putting the equation together:

$$Ku = f$$

$$\begin{bmatrix} 30 & 20 & -10 & 0 & 0 & 0 & -20 & -20 \\ & 20 & 0 & 0 & 0 & 0 & -20 & -20 \\ & & 10 & 0 & 0 & 0 & 0 & 0 \\ & & & 5 & 0 & -5 & 0 & 0 \\ & & & & 20 & 20 & -20 & -20 \\ & & & & & 25 & -20 & -20 \\ & & & & & & 40 & 40 \\ & & & & & & & 40 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Node 1 doesn't move, and node 2 only moves in the x direction therefore $u_{x1} = u_{y1} = u_{y2} = 0$

$$u = \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ 0 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix}$$

Therefore, the reduced matrix equation will be the following:

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 20 & 20 & -20 & -20 \\ 0 & 20 & 25 & -20 & -20 \\ 0 & -20 & -20 & 40 & 40 \\ 0 & -20 & -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

After the reduction, the matrix is still singular (column4=column5), therefore the equation is still not solvable.

This is due to the fact that a node is placed in the middle of a bar, with no restrictions on that node.

Nadim Saridar

Classwork 1

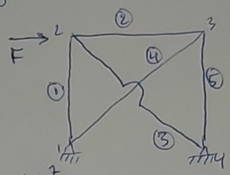
$E = 200\ \text{GPa} = 2\times10^{11}\ \text{Pa}$

$A = 6\ \text{cm}^2 = 6\times10^{-4}\ \text{m}^2$

Angles: ① 90° ② 0° ③ −45° ④ 45° ⑤ −90°

Elements' Lengths: ①②⑤ $L = 6\ \text{m}$

③④ $L = 6\sqrt{2}\ \text{m}$



~~4 elements~~ 4 nodes – 8 DoF

$$K^e = \left(\frac{EA}{L}\right)^e = \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

$$\left(\frac{EA}{L}\right)^{1,2,5} = 2\times10^7 \qquad \left(\frac{EA}{L}\right)^{3,4} = \sqrt{2}\times10^7$$

$$K^1 = K^5 = 2\times10^7 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad K^2 = 2\times10^7 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^3 = \sqrt{2}\times10^7 \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \qquad K^4 = \sqrt{2}\times10^7 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$K = 2\times10^7 \begin{bmatrix} 0.3536 & 0.3536 & 0 & 0 & -0.3536 & -0.3536 & 0 & 0 \\ & 1.3536 & 0 & -1 & -0.3536 & -0.3536 & 0 & 0 \\ & & 1.3536 & -0.3536 & -1 & 0 & -0.3536 & 0.3536 \\ & & & 1.3536 & 0 & 0 & 0.3536 & -0.3536 \\ & & & & 1.3536 & 0.3536 & 0 & 0 \\ & & & & & 1.3536 & 0 & -1 \\ & \text{symm.} & & & & & 0 & 0 \\ & & & & & & & 1 \end{bmatrix}$$

Each node has displacements in x and y, with nodes 1 and 4 that don't move, the displacement vector is the following:

$$u = \begin{bmatrix} 0 \\ 0 \\ u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \\ 0 \\ 0 \end{bmatrix}$$

Having one force in the x direction at node 2

$F = 80\ KN = 8 \times 10^4\ N$

$$f = \begin{bmatrix} 0 \\ 0 \\ 8\times10^4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The 8 equations will be reduced to 4:

$$2\times10^7 \begin{bmatrix} 1.3536 & -0.3536 & -1 & 0 \\ -0.3536 & 1.3536 & 0 & 0 \\ -1 & 0 & 1.3536 & 0.3536 \\ 0 & 0 & 0.3536 & 1.3536 \end{bmatrix} \begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} 8\times10^4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above equations:

$$\begin{bmatrix} u_{x_2} \\ u_{y_2} \\ u_{x_3} \\ u_{y_3} \end{bmatrix} = \begin{bmatrix} 8.54 \\ 2.23 \\ 6.77 \\ -1.77 \end{bmatrix} mm$$