Universitat Politècnica de Catalunya

Computational Solid Mechanics and Dynamics

Master's Degree in Numerical Methods in Engineering

Dynamics

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1 Solid and Structural Dynamics

1.1 In the dynamic system of slide 6, let r(t) be a constant force F. What is the effect of F on the time-dependent displacement u(t) and the natural frequency of vibration of the system?

This is a second-order linear ordinary differential equation. The equation will be.

$$F - ku = mu'' \longrightarrow \frac{d^2u}{dt^2} + \frac{k}{m}x = \frac{F}{m} \tag{1}$$

According to [1], the solution of this differential linear equation will be the sum of a particular solution and a general solution. The general solution of the homogeneous equation is an harmonic movement and will have the form of an oscillatory function whose phase and amplitude are determined by initial conditions, that is,

$$x_h(t) = A_h cos(w_0 t + \alpha) \tag{2}$$

As a particular solution, since the external force is of no oscillatory kind, it won't have an harmonic behavior, so it will be a constant in time depending only on F and k.

The representation of the proposed problem, considering that the movement is produced by a forced oscillation on a system which has viscous forces added can be seen in Fig. 1.

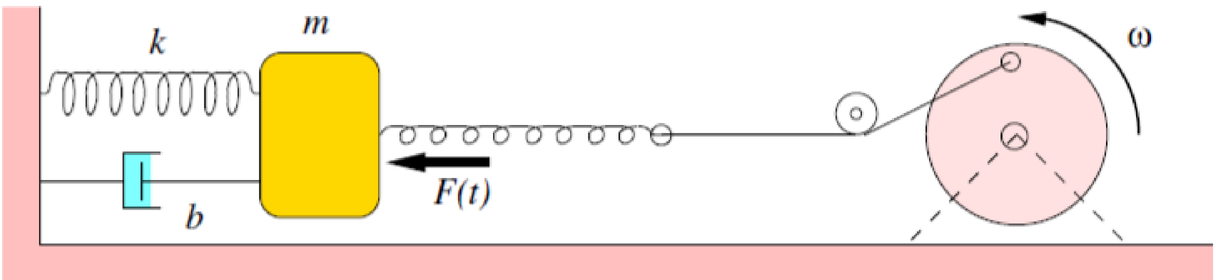

Figure 1: Representation of the system. Extracted from [1].

In this case, ω and b are simply zero, so the proposed solution to the displacement of the mass according to [1] will be

$$x(t) = A_h cos(\omega_0 t + \alpha) + \frac{F_0}{\omega_0^2 m} = A_h cos(\omega_0 t + \alpha) + \frac{F}{k} \tag{3}$$

since the natural frequency of the oscillator will only depend on m, k through $\omega_0 = \sqrt{k/m}$. This is already giving the answer to weather F affects the natural frequency. The answer is no, as it is only influenced by the spring and the mass and not the rest of the parameters is Fig. 1.

Then, from (3), where A_h, α are two parameters that will be determined based on the initial conditions of $x(t = 0)$ and its first derivative, it is clear that the force will contribute linearly to the increase of the displacements.

1.2 A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m, L, E and A . Suggestion: First determine the effective k.

From strength of materials, the deflection of the fixed-fixed beam proposed due to a static central load $P = mg$ is given by

$$\delta(x) = \frac{Fx^2}{48EI}(3L - 4x); \quad 0 \leq x \leq L/2 \tag{4}$$

Where I is the moment of inertia of the cross section. The natural frequency of the oscillation will be approximated by the same expression as in the previous exercise, already taking into account the effective stiffness $k = F/\delta = mg/\delta$ [1],

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta}} \tag{5}$$

The total deflection of the mass will be given by the deflection at the location of the mass due to the weight of the mass. Substituting thus taking into account that x = $L/2$:

$$\delta = \delta_{max} = \frac{L^3 mg}{192EI} \tag{6}$$

Thus an approximation to the natural frequency of the oscillation will be

$$\omega = \sqrt{\frac{192EI}{mL^3}} \tag{7}$$

If the cross-section of the bar is a square, then $I = A^2/12$ and

$$\omega = \sqrt{\frac{16EA^2}{mL^3}} = 4A\sqrt{\frac{E}{mL^3}} \tag{8}$$

1.3 Use the expression on slide 18 to derive the mass matrix of slide 17.

If the same shape functions used in the derivation of the stiffness matrix are chosen, the matrix is called the consistent mass matrix. It is denoted here by $\mathbf{M_C^e}$. For the 2-node prismatic bar element moving along x, the stiffness shape functions are $N_i = 1 - \frac{x-x_i}{l} = \frac{1-\xi}{2}$ and $N_j = \frac{x-x_i}{l} = \frac{1+\xi}{2}$. With $dx = ld\xi$, the consistent mass is easily obtained as

$$\mathbf{M_C^e} = \int_0^l \rho A(\mathbf{N_e})^T \mathbf{N_e} dx = \rho A \int_{-1}^{1} \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} d\xi = \frac{1}{6} \rho A l \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{9}$$

which is the matrix that we were looking for.

1.4 Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A1 to A2.

Now we are asked to derive the consistent mass matrix for a 2-node tapered bar element of length l and constant mass density ρ, moving along its axis x, if the cross section area varies as $A(\xi) = \frac{1}{2}A_1(1-\xi) + \frac{1}{2}A_2(1+\xi)$. Now it is simply required to introduce this expression into equation 10. With this it is obtained

$$\mathbf{M_C^e} = \int_0^l \rho A(\mathbf{N_e})^T \mathbf{N_e} dx = \rho \int_{-1}^{1} \left(\frac{1}{2}A_1(1-\xi) + \frac{1}{2}A_2(1+\xi) \right) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} d\xi \tag{10}$$

$$\mathbf{M_C^e} = \frac{\rho l}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & 3A_2 + A_1 \end{bmatrix} \tag{11}$$

1.5 A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

Globalization to 2D and 3D involves application of 2 × 4 and 2×6 transformation matrices, respectively. As the local element has zero stiffness in some directions, if the associated degrees of freedom are explicitly kept in the local stiffness, those rows and columns are zero and have no effect on the global stiffness. On the other hand, translational masses never vanish, so all translational masses must be retained in the local mass matrix. The two-node prismatic bar moving in the x, y plane has a diagonalized lumping matrix of the form

$$\mathbf{M_L}^e = \frac{1}{2} \rho A L \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

1.6 Concluding remarks

It has been studied that to do dynamic and vibration finite element analysis, it is needed a mass matrix as well as the stiffness matrix. Mass matrices for individual elements are formed in local coordinates, transformed to global, and merged into the master mass matrix following exactly the same techniques used for K. However, a notable difference with the stiffness matrix is the possibility of using a diagonal mass matrix based on direct lumping, and it entails significant computational advantages for calculations.

References

[1] Calaf Zayas, Jaume *Oscil·lacions : teoria i problemes.*, 2012 ISBN: 9788476539163 9788476539101