

ASSIGNMENT 10, COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

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Problem 1

- In the dynamic system of slide 6, let $r(t)$ be a constant force F . What is the effect of F on the time-dependent displacement $u(t)$ and the natural frequency of vibration of the system?

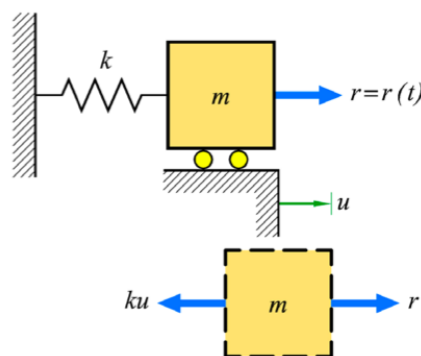


Figure 1: The dynamic system found in slide 6

By using the Newton's second law $F = ma$, we can obtain the differential equation representing the harmonic system above:

$$ku + m\ddot{u} = F \quad (1)$$

Given that this is a nonhomogeneous ODE, the solution will contain both a homogeneous and a particular solution term.

$$u(t) = u_h(t) + u_p(t) \quad (2)$$

The homogeneous solution is described by

$$u_h(t) = A\sin(\omega t) + B\cos(\omega t), \quad \text{where } \omega = \sqrt{\frac{k}{m}} \quad (3)$$

Given that the force F is considered to be constant, and does not have any time dependency, the solution is easily solved as

$$u_p(t) = u_p = \frac{F}{k} \quad (4)$$

By using the solution from (3) and (4) in (2) we get

$$u(t) = A\sin(\omega t) + B\cos(\omega t) + \frac{F}{k} \quad (5)$$

From the dynamic system we know that at $t = 0$ the system is at rest. This gives $u(t = 0) = 0$ and $\dot{u}(t = 0) = 0$. By these conditions we can calculate the value of A and B .

$$u(t = 0) = 0 \implies B = -\frac{F}{k} \quad (6)$$

$$\dot{u}(t = 0) = 0 \implies A = 0 \quad (7)$$

The complete solution of the ODE is therefore given by

$$u(t) = \frac{F}{k}(1 - \sin(\omega t)) \quad (8)$$

From this solution we can see that the force F does not affect the natural vibration frequency of the system (the term ω does not contain F), but it does affect the amplitude of the system.

Problem 2

- A weight whose mass is m is placed at the middle of a uniform axial bar of length L that is clamped at both ends. The mass of the bar may be neglected. Estimate the natural frequency of vibration in terms of m , L , E and A . Suggestion: First determine the effective k .

The bar clamped at both sides can be considered as a structural system with one DOF. The degree of freedom is the vertical displacement found at the middle of the bar (underneath the weight). Considering $F = mg$, E to be the Youngs Modulus, A to be square ($b = h$) and the inertia $I = \frac{bh^3}{12} = \frac{A^2}{12}$. The displacement at the middle of a beam with fixed supports subjected to a pointload is given by the beam formulas as

$$\delta = \frac{FL^3}{192EI} = \frac{FL^3}{16EA^2} = \frac{mgL^3}{16EA^2} \quad (9)$$

By using the system equation $Ku = F$, where $F = mg$ and $u = \delta$ we get

$$K \frac{mgL^3}{16EA^2} = mg \implies K = \frac{16EA^2}{L^3} \quad (10)$$

From this we can calculate the natural frequency of vibration in terms of m , L , E , and A .

$$\omega = \sqrt{\frac{K}{m}} = \frac{4A}{L} \sqrt{\frac{E}{mL}} \quad (11)$$

Problem 3

-Use the expression on slide 18 to derive the mass matrix of slide 17.

This basically means I'm supposed to use the term $\mathbf{m} = \int_L \mathbf{N}^T \mathbf{N} \rho dV$ to derive the matrix

$$\mathbf{m} = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}$$

\mathbf{N} is the vector containing the shape functions of the element. For a two-noded element we have

$$N_1 = 1 - \xi, N_2 = \xi \implies \mathbf{N} = \begin{bmatrix} 1 - \xi & \xi \end{bmatrix} \quad (12)$$

Further the integral can be written as

$$\mathbf{m} = \rho A \int_L \begin{bmatrix} (1 - \xi)^2 & (1 - \xi)\xi \\ \xi(1 - \xi) & (1 - \xi)^2 \end{bmatrix} dx \quad (13)$$

By substituting $dx = Ld\xi$ we get

$$\mathbf{m} = \rho AL \int_0^1 \begin{bmatrix} (1 - \xi)^2 & (1 - \xi)\xi \\ \xi(1 - \xi) & (1 - \xi)^2 \end{bmatrix} d\xi \quad (14)$$

The result of the integral gives the solution of \mathbf{m}

$$\mathbf{m} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (15)$$

Problem 4

- Obtain also the mass matrix of a two-node, linear displacement element with a variable cross-sectional area that varies from A_1 to A_2 .

The expression of A can be given as

$$A(\xi) = A_1(1 - \xi) + A_2(\xi) = A_1N_1 + A_2N_2 \quad (16)$$

Due to the fact that A will differ from $\xi = 0$ and $\xi = 1$, the expression of A can not be subtracted from the integral

$$\mathbf{m} = \rho L \int_0^1 \begin{bmatrix} (1-\xi)^2 & (1-\xi)\xi \\ \xi(1-\xi) & (1-\xi)^2 \end{bmatrix} (A_1(1-\xi) + A_2\xi) d\xi \quad (17)$$

It is simplified by dividing the expression into two integrals

$$\mathbf{m} = A_1\rho L \int_0^1 \begin{bmatrix} (1-\xi)^2 & (1-\xi)\xi \\ \xi(1-\xi) & (1-\xi)^2 \end{bmatrix} (1-\xi) d\xi + A_2\rho L \int_0^1 \begin{bmatrix} (1-\xi)^2 & (1-\xi)\xi \\ \xi(1-\xi) & (1-\xi)^2 \end{bmatrix} \xi d\xi \quad (18)$$

The result of these integrals gives

$$\mathbf{m} = A_1\rho L \begin{bmatrix} \frac{1}{4} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} \end{bmatrix} + A_2\rho L \begin{bmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{4} \end{bmatrix} \quad (19)$$

Which gives the total expression of

$$\mathbf{m} = \frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix} \quad (20)$$

If the case is $A_1 = A_2 = A$, it is easy to see that \mathbf{m} is equal to the matrix obtained in the previous problem.

Problem 5

- A uniform two-node bar element is allowed to move in a 3D space. The nodes have only translational d.o.f. What is the diagonal mass matrix of the element?

For a two-node bar element allowed to translation in a 3D space, the number of dofs will be 6. Each degree of freedom represent displacement in one of the three directions for each of the two nodes. Due to the fact that the element is uniform, the diagonal mass matrix can be addressed as the following lumped matrix

$$\mathbf{m} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \cdot & 1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ sym & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad (21)$$