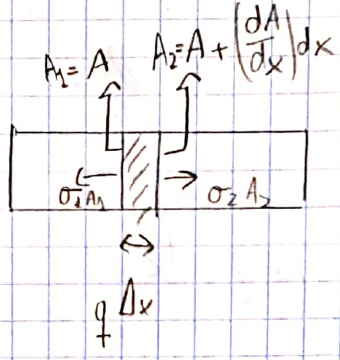
Assignment 2.3 Extra

$$A = A_i (1 - \xi) + A_j \xi$$

a) Stiffness matrix




* Balance on $\Delta x$:

$$-\sigma_1 A_1 + \sigma_2 A_2 + q \Delta x = 0$$

* $$\sigma = E \frac{du}{dx}$$

* Strong form:

$$-EA \frac{d^2u}{dx^2} - E \frac{du}{dx} \frac{dA}{dx} = q$$

* To obtain weak form of the problem:

$$-E \int_0^{\ell} WA \frac{d^2u}{dx^2} dx - E \int_0^{\ell} W \frac{du}{dx} \frac{dA}{dx} dx = \int_0^{\ell} W q \, dx$$

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+ Integrating and using divergence theorem, it is obtained:

$$-E\int_0^l A\,\frac{dW}{dx}\,\frac{du}{dx} + \int_0^l Wq\,dx + \left[WA\,\frac{du}{dx}\right]_0^l = 0$$

+ As there is not a prescribed flux:

$$E\int_0^l A\,\frac{dW}{dx}\,\frac{du}{dx} = \int_0^l Wq\,dx$$

+ approximation of u

$$u(x) = \sum_{j=1}^{n} N_j(x)\,u(x_j)$$

n = order of discretisation

+ Galerkin form

$$W_i = N_i$$

+ Substituting:

$$\left(E\int_0^l A\,\frac{dN_i}{dx}\,\frac{dN_j}{dx}\,dx\right)u_j = \int_0^l q\,N_i\,dx \qquad i,j = 1\ldots n$$

+ $\left[K\cdot u = f\right]$

where,

$$K_{ij} = \frac{E}{l}\int_0^1 \left[(1-\xi_1)A_1 + \xi A_2\right]\frac{dN_i}{d\xi}\cdot\frac{dN_j}{d\xi}\,d\xi$$

$$f_i = l \int_0^1 q \, N_i \, d\xi$$

* Shape function for linear elements

$N_1(\xi) = 1 - \xi \qquad N_2(\xi) = \xi$

* Stiffness matrix:

$$K = \frac{E}{l} \frac{A_1 + A_2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

* If $A = \frac{1}{2}(A_i + A_j)$

$$K = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \text{Same solution}$$

b) Load vector $f^{(e)}$ → force $q = \rho g A(\xi)$

$q(x) = \rho g A(\xi)$

$$F = \frac{\rho g h}{6} \begin{bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{bmatrix}$$

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* If $A_1 = A_2 = A$ (constant area)

$$F = \frac{\rho g h A}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

* If $A_1 \neq A_2 = 0$,

$$F = \frac{\rho g h A_1}{6} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

c) load vector $\rightarrow$ Force $Q$ at $x = a$

$$q(x) = Q\,\delta(x-a)$$

$$F_i = Q \int_0^{\ell} N_i\, \delta(x-a)\, dx \quad ; \quad \int_0^{\ell} f(x)\, \delta(x-a)\, dx = f(a)$$

So,

$$F_i = Q N_i \left( \frac{a - x_i}{\ell} \right)$$

$$\left[ F_i = Q \begin{bmatrix} 1 - a/\ell \\ a/\ell \end{bmatrix} \right]$$

* For $a = 0$ $\rightarrow$ External load only at first node
* For $a = \ell$ $\Rightarrow$ External load only at second node
* For $a = \frac{h}{2}$ $\Rightarrow$ Same force vector $q' = \frac{Q}{\ell} x$