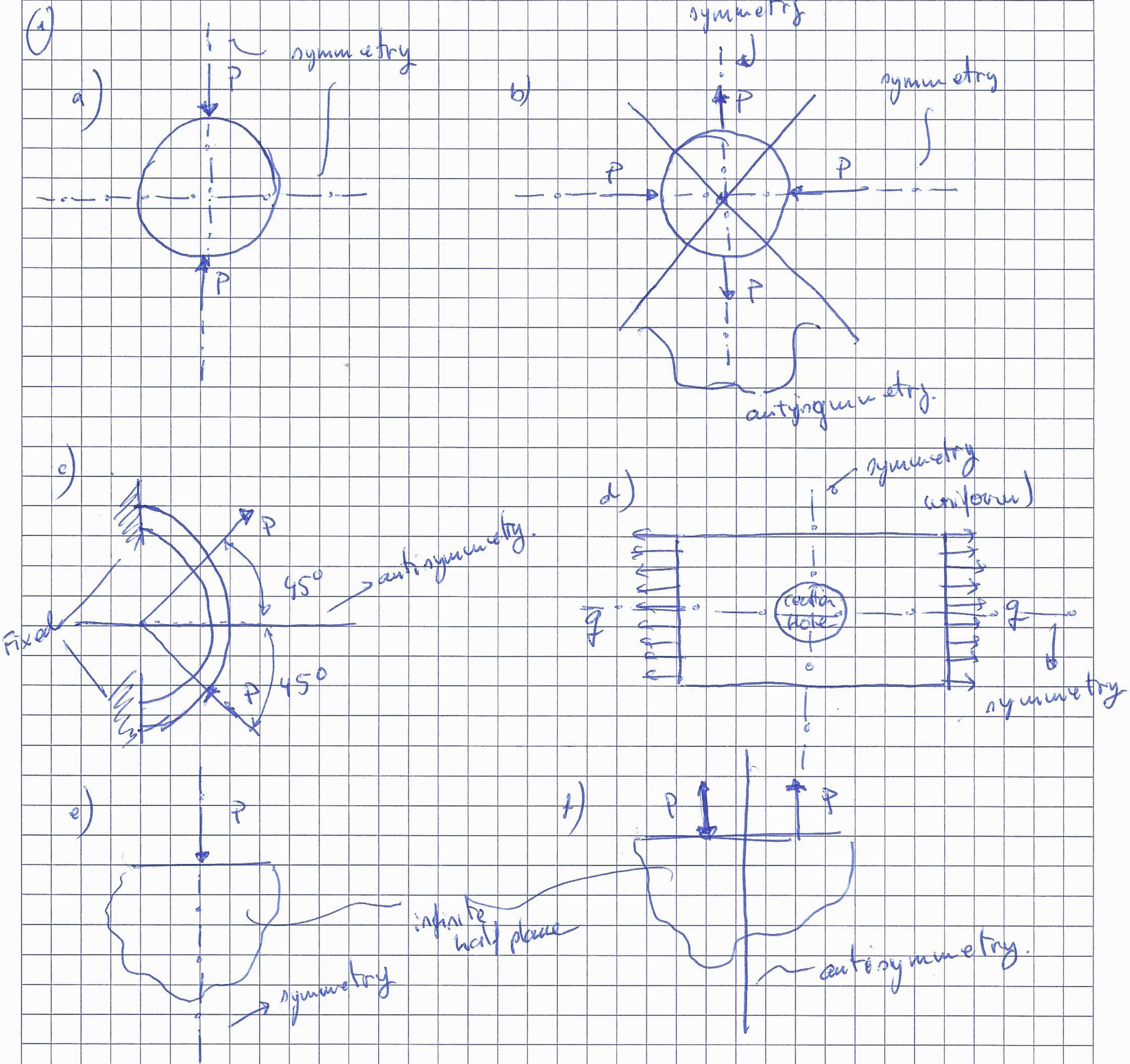
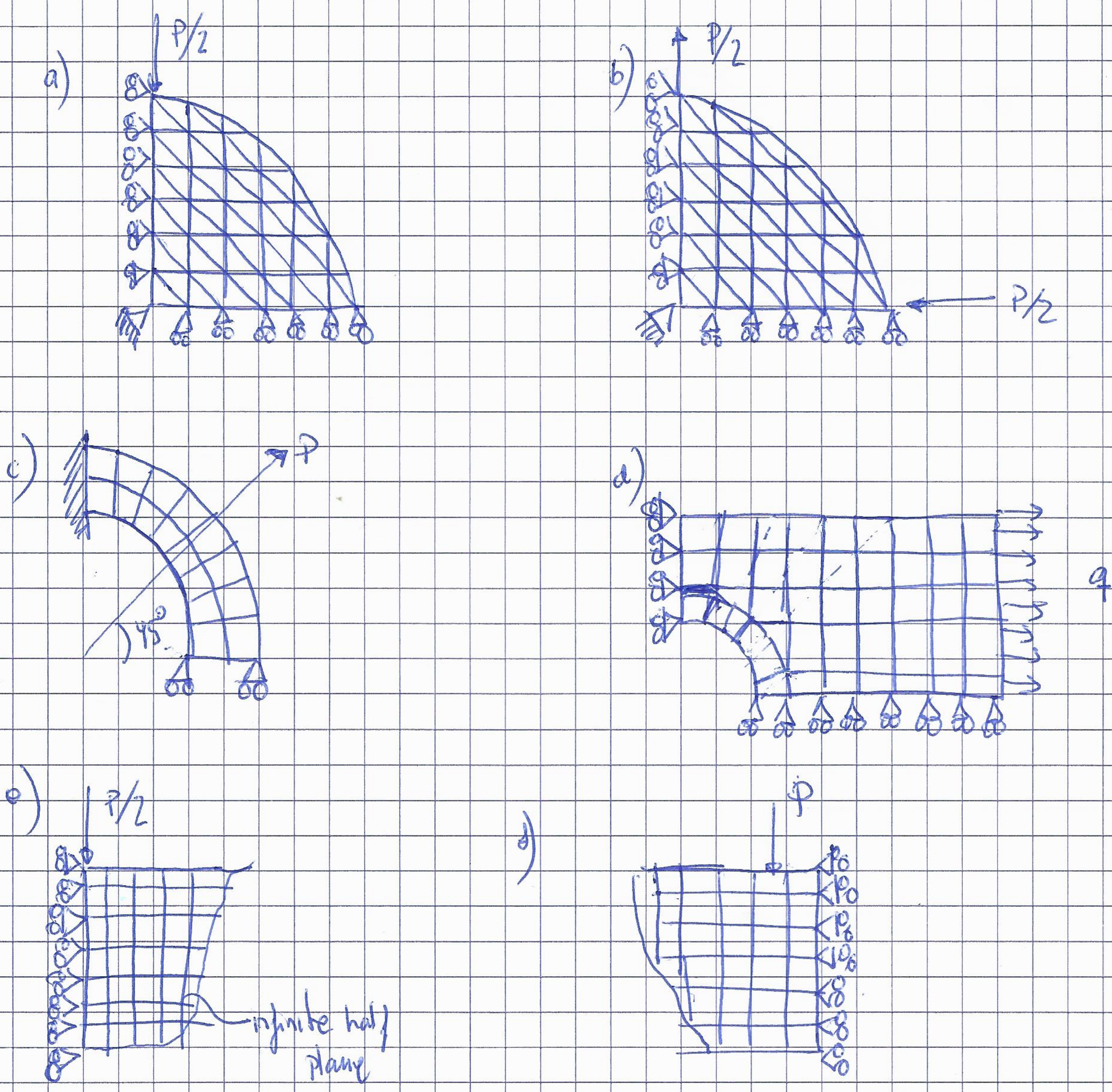
24/02/2020

Assignment 2.1

1)





Assignment 2.2

verification: guarantees that the answer converged for the finite element solution of the model.

Validation : shows that the model is truly representative of the problem we have at hand.

thus, the FE solution must be verified and the model must be validated in order to ensure a reliable analysis. in other words; validation is the assessment of the accuracy of both the structural and computational models by comparison of the numerical results with experimental data.

Assignment 2.2 (cont)

The correct definition of the experimental tests and the reliability of the experimental results are crucial issues in the validation process.

On the other hand, in verification, the relationships between the numerical results to the real world is not an issue.
The verification of FEM computations is tipically made by comparing the numerical results for simple benchmark problems with exact solutions obtained analytically, or using more accurate numerical methods.
Verification is usually performed first in order to evaluate and reduce the possible sources of numerical errors.
Summarizing, verification serves to check that we are solving ~~pr~~ structural problems accurately, while validation tell us that we are solving the right problem.

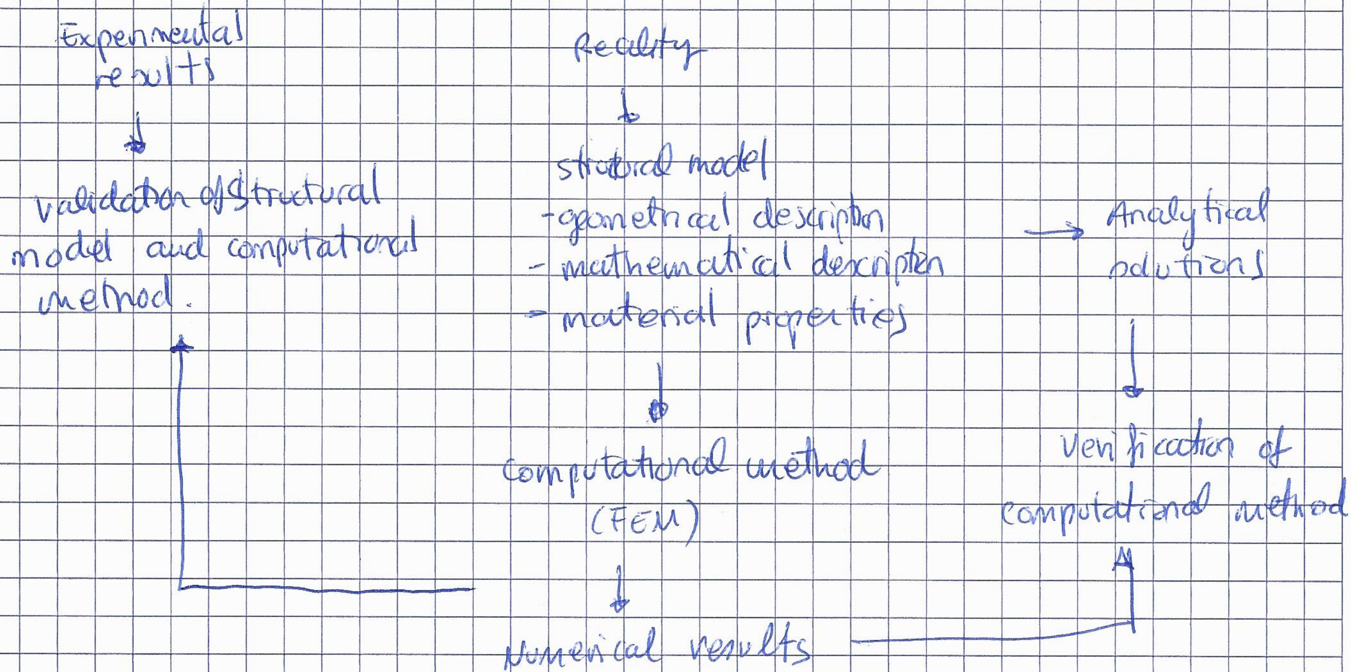
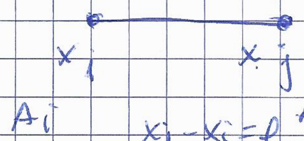


Figure of verification and validation processes in FEM.

Assignment 2.3

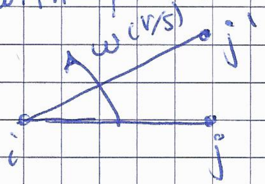
1)



$x_j - x_i = \ell$

$A = A_i(1-\xi) + A_j\xi$

with $\rho \rightarrow$ cte



$q(x) = \rho A \omega^2 \cancel{x}$ $\Longrightarrow$ consistent node forces are written as:

$$f = \int_0^1 q \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \ell d\xi$$

since we have $q(x)$ in terms of x, we have to express it as $q(\xi)$ given the fact that: $\xi = \frac{x - x_\ell}{\ell}$ or $\xi = \frac{x - x_i}{x_j - x_i}$ and since $x_j - x_i = \ell$

thus, we have $\xi = \frac{x}{\ell} \Rightarrow x = \ell\xi$ $\quad$ $i \Rightarrow \begin{cases} x_i = 0 \\ x_j = \ell \end{cases}$

and, by contruction, $q(x) \longrightarrow q(\xi) = \rho A \omega^2 \ell \xi$

and, so,

$$f = \int_0^1 q(\xi) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \ell d\xi = \int_0^1 \rho A \omega^2 \ell^2 \xi \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} d\xi$$

also, we know $\rightarrow$ $A(\xi) = A_i(1-\xi) + A_j\xi$

$$\hookrightarrow f = \int_0^1 \rho \ell^2 \omega^2 \xi \left(A_i(1-\xi) + A_j\xi\right) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} d\xi =$$

$$= \rho\omega^2\ell^2 \int_0^1 \begin{bmatrix} (A_i(1-\xi) + A_j\xi)(1-\xi)\xi \\ (A_i(1-\xi) + A_j\xi)\xi^2 \end{bmatrix} d\xi =$$

$$= \rho\omega^2\ell^2 \int_0^1 \begin{bmatrix} A_i(\xi + \xi^3 - 2\xi^2) + A_j(\xi^2 - \xi^3) \\ A_i(\xi^2 - \xi^3) + A_j\xi^3 \end{bmatrix} d\xi$$

$$= \rho\omega^2 \ell^2 \begin{bmatrix} A_i\left(\xi^2/2 - 2/3\,\xi^3 + \xi^4/4\right) + A_j\left(\xi^3/3 - \xi^4/4\right) \\ A_i\left(\xi^3/3 - \xi^4/4\right) + A_j\,\xi^4/4 \end{bmatrix}_0^1$$

$$= \rho\omega^2 \ell^2 \begin{bmatrix} A_i\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right) + A_j\left(\frac{1}{3} - \frac{1}{4}\right) \\ A_i\left(\frac{1}{3} - \frac{1}{4}\right) + A_j\,\frac{1}{4} \end{bmatrix}$$

$$f = \rho\omega^2 \cdot \ell^2 \begin{bmatrix} A_i\,\frac{1}{12} + A_j\,\frac{1}{12} \\ A_i\,\frac{1}{12} + A_j\,\frac{1}{4} \end{bmatrix}$$

if $A = A_j = A_i$

$$f = \rho\omega^2 \ell^2 \begin{bmatrix} A \cdot \frac{1}{6} \\ A \cdot \frac{1}{3} \end{bmatrix}$$

Assignment 2.4

Tapered bar element $\rightarrow$ $A = A_i(1-\xi) + A_j\xi$

$x_j - x_i = \ell$

constant area bar $\Rightarrow$ $A = \frac{1}{2}(A_i + A_j)$

$\xi = \frac{x - \cancel{x}_\ell}{\ell} \Leftrightarrow \xi = \frac{x - x_i}{x_j - x_i}$

we have that the expression for the element stiffness matrix reads as:

if $x_i = 0$, $x_j = \ell$ $\Rightarrow$ $\xi = \frac{x}{\ell} \Rightarrow x = \xi\ell$

$$K_e = \int_0^\ell EA\, B^T B\, dx = \int_0^\ell \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

since we have $A(\xi)$ in terms of ξ, we have to express it in terms of x given the fact that $x = \xi\ell$:

$$A(\xi) \Rightarrow A(x) = A_i\left(1 - \frac{x}{\ell}\right) + A_j\frac{x}{\ell}$$

and so, the stiffness matrix expression,

$$K_e = \int_0^\ell \frac{E}{\ell^2}\left[A_i\left(1 - \frac{x}{\ell}\right) + A_j\frac{x}{\ell}\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

$$= \frac{E}{\ell^2}\left[-A_i\frac{x^2}{2\ell} + A_i x + A_j\frac{\cancel{x}^2}{2\ell}\right]_0^\ell \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{E}{\ell^2}\left[-A_i\frac{\ell}{2} + A_i\ell + A_j\frac{\ell}{2}\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\frac{E}{\ell}(A_i + A_j)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which stands the same expression for that of a constant area bar with a cross section $A = \frac{1}{2}(A_i + A_j)$

b)

bar of constant area A

subject to a force $q(\xi) = \rho g A(\xi)$ where $A(\xi) = A_i(1-\xi) + A_j(\xi)$

$\rho g \rightarrow$ cte

check 1) $A_i = A_j$

2) $A_j = 0$

$$f = \int_0^1 q(\xi) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \ell \, d\xi$$

$$f = \int_0^1 \rho g \left(A_i(1-\xi) + A_j \xi \right) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \ell \, d\xi$$

$$= \rho g \ell \int_0^1 \left(A_i(1-\xi) + A_j \xi \right) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} d\xi$$

$$= \rho g \ell \int_0^1 \left(A_i - A_i \xi + A_j \xi \right) \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} d\xi$$

$$= \rho g \ell \int_0^1 \begin{bmatrix} (A_i - A_i\xi + A_j\xi)(1-\xi) \\ (A_i - A_i\xi + A_j\xi)\xi \end{bmatrix} d\xi$$

$$= \rho g \ell \int_0^1 \begin{bmatrix} A_i - A_i\xi - A_i\xi + A_i\xi^2 + A_j\xi - A_j\xi^2 \\ A_i\xi - A_i\xi^2 + A_j\xi^2 \end{bmatrix} d\xi$$

$$= \rho g \ell \begin{bmatrix} A_i\xi - 2\frac{A_i}{2}\xi^2 + \frac{A_i}{3}\xi^3 + \frac{A_j}{2}\xi^2 - A_j\frac{\xi^3}{3} \\ \frac{A_i}{2}\xi^2 - \frac{A_i}{3}\xi^3 + \frac{A_j}{3}\xi^3 \end{bmatrix}_0^1$$

$$= \rho g \ell \begin{bmatrix} A_i\frac{1}{3} + A_j\frac{1}{2} - A_j\frac{1}{3} \\ A_i\frac{1}{2} - A_i\frac{1}{3} + A_j\frac{1}{3} \end{bmatrix} = \rho g \ell \begin{bmatrix} \frac{1}{3}A_i + \frac{1}{6}A_j \\ \frac{1}{6}A_i + \frac{1}{3}A_j \end{bmatrix}$$

1) if $A_i = A_j$

$$f = \rho g \ell \begin{bmatrix} \frac{1}{2}A_i \\ \frac{1}{2}A_i \end{bmatrix}$$

2) if $A_j = 0$

$$f = \rho g \ell \begin{bmatrix} \frac{1}{3}A_i \\ \frac{1}{6}A_i \end{bmatrix}$$

e)

$g(x) = Q\,\delta(x-a) \Rightarrow \delta(x-a) \Rightarrow$ 1D Dirac's Delta at $x=a$

$$f = \int_0^1 q(\xi)\begin{bmatrix}1-\xi\\ \xi\end{bmatrix}\ell\, d\xi \qquad \xi = \frac{x}{\ell} \Rightarrow x = \xi\ell$$

$$q(x) \Rightarrow q(\xi) = Q\,\delta(\xi\ell - a)$$

$$f = \int_0^1 Q\,\delta(\xi\ell - a)\begin{bmatrix}1-\xi\\ \xi\end{bmatrix}\ell\, d\xi \quad \text{or} \quad f = \int_0^\ell q(x)\begin{bmatrix}1-\frac{x}{\ell}\\ \frac{x}{\ell}\end{bmatrix}dx$$

↳ we need to satisfy the condition $x - a = 0$ such that $\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$

↳ if the bounds of the integral are not infinity, we need to make sure that if we let $x=a$, a would be in the bounds of the integral or else the integral would evaluate into zero. i.e.

$$\int_{-b}^{b} f(x)\,\delta(x-a)\,dx = \begin{cases} 0, & \text{if } b < a \text{ or } -b > a \text{ s.t. we cannot let } x \neq a \\ f(0), & \text{if } -b < a < b \text{ s.t. we can let } x = a \end{cases}$$

thus,

$$f = \int_0^\ell Q\,\delta(x-a)\begin{bmatrix}1-\frac{x}{\ell}\\ \frac{x}{\ell}\end{bmatrix}dx = \begin{cases} 0 & \text{if } \ell < a \text{ or } a < 0 \text{ s.t. we cannot let } x \neq a \\ Q\begin{bmatrix}1-\frac{a}{\ell}\\ \frac{a}{\ell}\end{bmatrix} & \text{if } 0 < a < b \text{ s.t. we can let } x = a \end{cases}$$