Classwork 2

Computational Structural Mechanics and Dynamics

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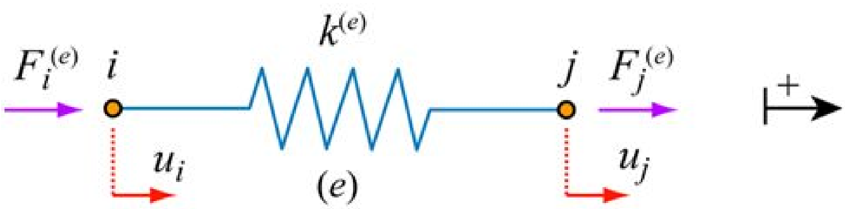

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1 Problem statement

On "Variational Formulation":

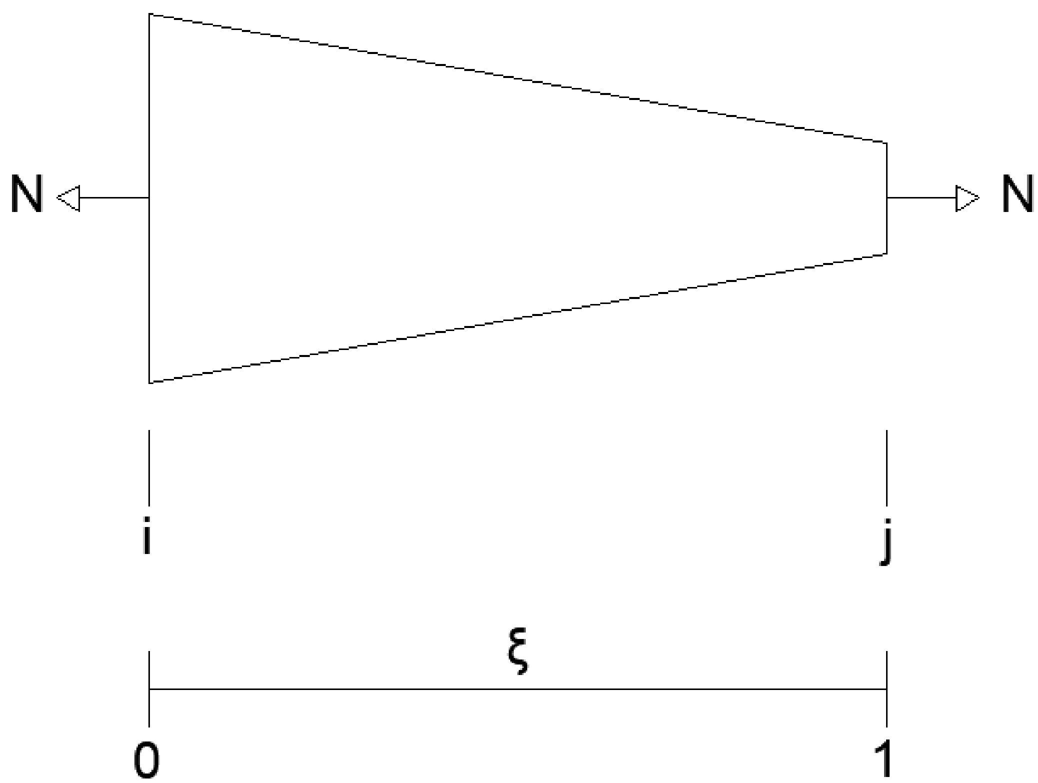
1. Derive the stiffness matrix for a tapered bar element in which the cross section area varies linearly along the element length:

$$A = A_i(1-\xi) + A_j\xi$$

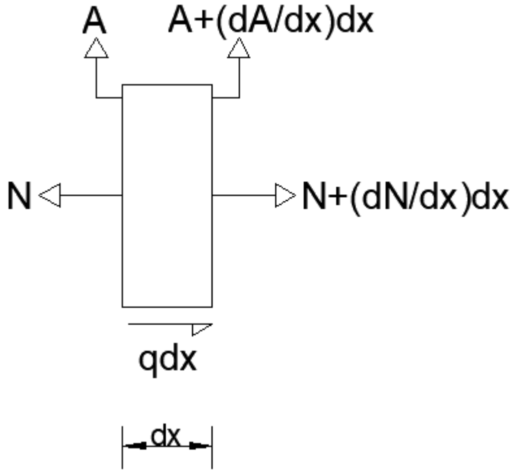
where A_i and A_j are the areas at the end nodes, and ξ is the natural dimensionless coordinate for a bar member. Show that yields to the same answer that of a stiffness of a constant area bar with cross section $A = \frac{1}{2}(A_i + A_j)$.

2. Find the consistent load vector f^e for a bar of constant area A subject to a force $q = \rho g A(\xi)$ in which $A(\xi)$ varies according to question (1) and ρ,g are constants. Check the cases $A_i = A_j$, and $A_j = 0$.

3. Find the consistent load vector f^e if the bar is subjected to a concentrated axial force Q at a distance $x = a$ from its left end. Consider $q(x) = Q\delta(x–a)$ in which $\delta(x–a)$ is the onedimensional Dirac's delta function at $x = a$. Check the results for the relevant cases of (1).



1.1 Solution 1



$$-N + N + \frac{dN}{dx}dx + qdx = 0$$

$$\frac{dN}{dx}dx + qdx = 0$$

Where:

$$\sigma = \frac{N}{A}$$

Therefore:

$$N = \sigma A$$

Substituting:

$$\frac{d(\sigma A + \cancelto{0}{\frac{d\sigma A}{dx}dx})}{dx}dx + qdx = 0$$

$$\frac{d\sigma A}{dx} + q = 0$$

$$A\frac{d\sigma}{dx} + \sigma\frac{dA}{dx} + q = 0$$

Where:

$$\sigma = \varepsilon E$$

$$\varepsilon = \frac{du}{dx}$$

Therefore:

$$A\frac{d}{dx}(E\frac{du}{dx}) + E\frac{du}{dx}\frac{dA}{dx} + q = 0$$

Integrating the differential equation:

$$\int WA\frac{d}{dx}(E\frac{du}{dx})dx + \int WE\frac{du}{dx}\frac{dA}{dx}dx + \int Wqdx = 0$$

$$EA\int \frac{dW}{dx}\frac{du}{dx}dx = \int Wqdx + [EAW\frac{du}{dx}]_{x_i}^{x_j}$$

Where, the approximation of $u(x)$:

$$u(x) \approx u_h(x) = \sum N_j(x)u_j$$

And:

$$W_i = N_i$$

Therefore:

$$EA\int \frac{dN_i}{dx}\frac{dN_j}{dx}dx = \int N_i q dx + [EAN_i\frac{dN_i}{dx}]_{x_i}^{x_j}$$

Therefore the stiffness matrix is defined as follows:

$$K_{ij} = E\int_{x_1}^{x_2} A\frac{dN_i}{dx}\frac{dN_j}{dx}dx$$

$$K_{ij} = \frac{E}{h}\int_0^1 [A_i(1-\xi) + A_j\xi]\frac{dN_i}{d\xi}\frac{dN_j}{d\xi}d\xi$$

Where the shape functions are defined as follows:

$$N_1 = 1-\xi \quad => \quad \frac{dN_1}{d\xi} = -1$$

$$N_2 = \xi \quad => \quad \frac{dN_2}{d\xi} = 1$$

Thus:

$$K_{ij} = \frac{E}{h}\frac{A_i + A_j}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For a constant area $A = \frac{[A_i+A_j]}{2}$:

$$K_{ij} = E\frac{[A_i + A_j]}{2}\int_0^h \frac{dN_i}{dx}\frac{dN_j}{dx}dx$$

Where the shape functions are:

$$N_1 = \frac{x_2 - x}{h} \quad => \quad \frac{dN_1}{dx} = \frac{-1}{h}$$

$$N_2 = \frac{x - x_1}{h} \quad => \quad \frac{dN_2}{dx} = \frac{1}{h}$$

Therefore:

$$K_{ij} = \frac{E}{h}\frac{A_i + A_j}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

1.2 Solution 2

The force vector is defined as follows:

$$f_i = \int_{x_1}^{x_2} N_i q dx$$

$$f_i = \rho g h\int_0^1 [A_i(1-\xi) + A_j\xi]N_i d\xi$$

Therefore:

$$f = \frac{\rho g h}{6} \begin{bmatrix} 2A_i + A_j \\ A_i + 2A_j \end{bmatrix}$$

When $A = A_i = A_j$:

$$f = \frac{\rho g h A}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

And when $A_j = 0$:

$$f = \frac{\rho g h A_i}{6} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

1.3 Solution 3

Given:

$$q(x) = Q\delta(x-a)$$

The force vector is defined as:

$$f_i = Q \int_{x_1}^{x_2} N_i(x)\delta(x-a)dx$$

Where:

$$\int_{x_1}^{x_2} f(x)\delta(x-a)dx = f(a) \quad x_1 < a < x_2$$

Therefore:

$$f_i = Q \int_{x_1}^{x_2} N_i(x)\delta(x-a)dx = QN_i(a)$$

Where:

$$N_1 = 1 - \frac{a}{h}$$

$$N_2 = \frac{a}{h}$$

Therefore:

$$f = Q \begin{bmatrix} 1 - \frac{a}{h} \\ \frac{a}{h} \end{bmatrix}$$