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Computational Structural Mechanics and Dynamics - Assignment 2 (Extra)
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1 Problem Description

a) Derive the stiffness matrix for a tapered bar element in which the cross section area varies linearly along the element length:

$$A = A_i(1-\xi) + A_j\xi \tag{1}$$

Where A_i and A_j are the areas at the end nodes, and ξ is the natural dimensionless coordinate for a bar member. Show that yields to the same answer that of a stiffness of a constant area bar with cross section $A = \frac{1}{2}(A_i + A_j)$.

b) Find the consisten load vector f^e for a bar of constant area A subject to a force $q = \rho g A(\xi)$ in which $A(\xi)$ varies according to question a) and ρ, g are constants. Check the cases $A_i = A_j$ and $A_j = 0$.

c) Find the consisten load vector f^e if the bar is subjected to a concentrated axial force Q at a distance x = a from its left end. Consider $q(x) = Q\delta(x-a)$ in which $\delta(x-a)$ is the one dimensional Dirac's delta function at x = a. Check the results for the relevant cases of A.

2 Solution

a) The stiffness matrix of a 1D bar element structural problem is given by:

$$K = \int_{x_1}^{x_2} \frac{EA}{l}\frac{dN_i}{dx}\frac{dN_j}{dx}dx \qquad i,j = 1,2 \tag{2}$$

Where:
E is the Young's modulus of the material
A is the cross-section area of the element
N_i and N_j are shape functions

Since in this case, A is a function of ξ, which describes coordinates of a normalized domain [0,1], we must transform our domain in the following way:

$$1-\xi = \frac{l-x}{l} \quad \rightarrow \quad \xi = \frac{x}{l} \quad \rightarrow \quad \frac{d\xi}{dx} = \frac{1}{l} \tag{3}$$

Hence, the shape functions will take the form:

$$N_1 = 1-\xi \qquad N_2 = \xi \tag{4}$$

$$\frac{dN_1}{dx} = \frac{dN_1}{d\xi}\frac{d\xi}{dx} = -\frac{1}{l} \qquad \frac{dN_2}{dx} = \frac{dN_2}{d\xi}\frac{d\xi}{dx} = \frac{1}{l} \tag{5}$$

Therefore, the stiffness matrix becomes:

$$K = \frac{E}{l}\frac{dN_i}{dx}\frac{dN_j}{dx}\int_0^1 A_i(1-\xi) + A_j\xi dx = \frac{E}{l}\frac{dN_i}{dx}\left[A_i(\xi - \frac{\xi^2}{2}) + A_j\frac{\xi^2}{2}\right]_0^1 \quad (6)$$

$$\boxed{K = \frac{E}{2l}(A_i + A_j)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \quad (7)$$

It may be observed that this poses the same result as taking the area as constant with the average value between A_i and A_j.

b) The consistent nodal force vector for a 1D bar element is given by:

$$f_{ext} = \int_0^1 q\begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} ld\xi \quad (8)$$

Where:

$\xi = \frac{x-x_1}{l} = \frac{x}{l}$ (in this case, $x_1 = 0$)

$q = \rho g A(\xi)$

$A = A_i(1-\xi) + A_j\xi$

Replacing these values in equation (8), we obtain the following expression:

$$f_{ext} = \rho g l\int_0^1 \begin{bmatrix} A_i(\xi - 2\xi + \xi^2) + A_j(\xi - \xi^2) \\ A_i(\xi - \xi^2) + A_j\xi^2 \end{bmatrix} d\xi \quad (9)$$

Integrating yields:

$$\boxed{f_{ext} = \frac{\rho g l}{6}\begin{bmatrix} 2A_i + A_j \\ A_i + 2A_j \end{bmatrix}} \quad (10)$$

Now, we will particularize the solution for two cases:

- $A_i = A_j$

$$f_{ext} = \frac{\rho g l A}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (11)$$

- $A_j = 0$

$$f_{ext} = \frac{\rho g l A_i}{6}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (12)$$

c) In this case, q(x) is defined by $\delta(x-a)$, which means that it will only be non zero when x = a. By transforming the Dirac's delta function into the normalized domain, we get:

$$\delta(x-a) = \delta(\xi - \frac{a}{l}) \quad (13)$$

$$q(\xi) = Q\delta(\xi - \frac{a}{l}) \tag{14}$$

Replacing $q(\xi)$ into equation (8) yields:

$$f_{ext} = Ql\int_0^1 \delta(\xi - \frac{a}{l}) \begin{bmatrix} 1-\xi \\ \\ \xi \end{bmatrix} d\xi = Ql\int_0^1 \begin{bmatrix} 1-\frac{a}{l} \\ \\ \frac{a}{l} \end{bmatrix} d\xi = \boxed{Q \begin{bmatrix} l-a \\ \\ a \end{bmatrix}} \tag{15}$$