Assignment 3

Computational Structural Mechanics and Dynamics

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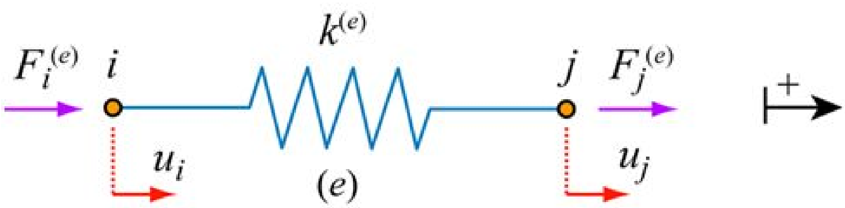


Contents

On "Plane stress problem" and "Linear Triangle"

1 Assignment 3.1

1. Compute the entries of K^e for the following plane stress triangle:

$$x_1 = 0; \quad y_1 = 0; \quad x_2 = 3; \quad y_2 = 1; \quad x_3 = 2; \quad y_3 = 2$$

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \; h = 1$$

   Partial result: $K_{11} = 18,75,\; K_{66} = 118,75$.

2. Show that the sum of the rows (and columns) 1, 3 and 5 of K^e as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

1.1 Solution

Since $h = 1 = constant$, we are facing a plane stress problem where the stiffness matrix is defined by the following expression:

$$K^e = B^T E B$$

$$K^e = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Where:

$$x_{jk} = x_j - x_k \quad y_{jk} = y_j - y_k$$

The area can be obtained as follows:

$$A = \frac{1}{2} det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\right) = 2$$

Obtaining the B matrix and substituting A and E into K^e, it yields as follows:

$$K^e = \frac{1}{4(2)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$K^e = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125 \\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25 \\ -12.5 & 6.25 & 75 & -37.5 & -62.5 & 31.25 \\ -6.25 & 12.5 & -37.5 & 75 & 43.75 & -87.5 \\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125 \\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix}$$

1.2 Solution

To show the 2nd statement of the problem, from the first row, we are going to sum the 1,3 and 5 stiffness:

$$18.75 - 12.5 - 6.25 = 0$$

And know the 2,4 and 6:

$$9.375 - 6.25 - 3.125 = 0$$

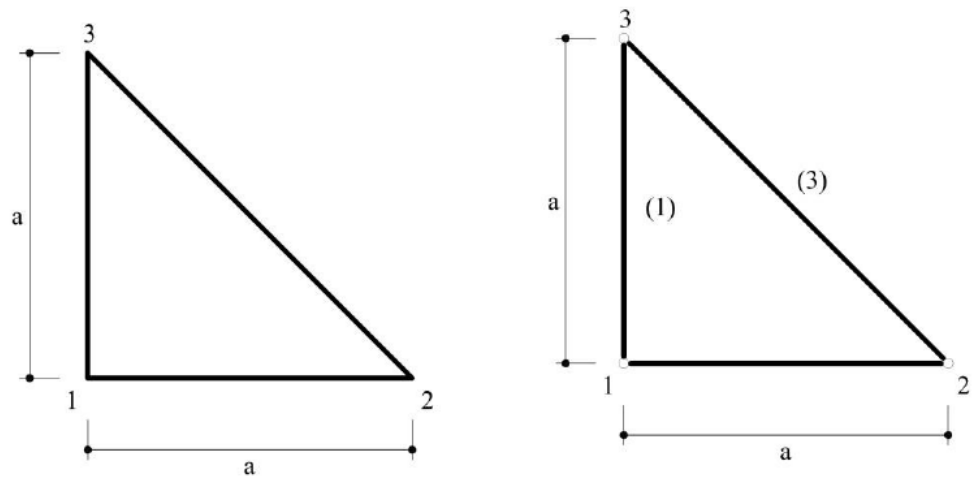
1.3 Conclusion

This occurs due to the following principles:

- Compatibility: The joint displacements of all members meeting at a joint must be the same.
- Equilibrium: The sum of forces exerted by all members that meet at a joint must balance the external force applied to that joint.

2 Assignment 3.2

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E,ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:



- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

1. Calculate the stiffness matrix K^e for both models.
2. Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

3. Why these two stiffness matrix are not equivalent? Find a physical explanation.
4. Solve question 1) considering $\nu \neq 0$ and extract some conclusions.

2.1 Solution

2.1.1 Turner triangle

$$K^e = B^T E B$$

$$K^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Substituting:

$$K^e = \frac{1}{4(A)}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K^e = \frac{E}{4A}\begin{bmatrix} 1.5 & 0.5 & -1 & -0.5 & -0.5 & 0 \\ 0.5 & 1.5 & 0 & -0.5 & -0.5 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.1.2 Three bar

Element 1

$$K^1 = EA_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} ux1 \\ uy1 \\ ux2 \\ uy2 \end{bmatrix}$$

Element 2

$$K^2 = EA_2 \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} ux2 \\ uy2 \\ ux3 \\ uy3 \end{bmatrix}$$

Element 3

$$K^3 = EA_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} ux1 \\ uy1 \\ ux3 \\ uy3 \end{bmatrix}$$

Global stiffness matrix:

$$K = \frac{E}{4}\begin{bmatrix} 4A_1 & 0 & -4A_1 & 0 & 0 & 0 \\ 0 & 4A_3 & 0 & 0 & 0 & -4A_3 \\ -4A_1 & 0 & 4A_1 + A_2\sqrt{2} & -A_2\frac{\sqrt{2}}{4} & -A_2\frac{\sqrt{2}}{4} & A_2\frac{\sqrt{2}}{4} \\ 0 & 0 & -A_2\frac{\sqrt{2}}{4} & A_2\frac{\sqrt{2}}{4} & A_2\frac{\sqrt{2}}{4} & -A_2\frac{\sqrt{2}}{4} \\ 0 & 0 & -A_2\frac{\sqrt{2}}{4} & A_2\frac{\sqrt{2}}{4} & A_2\frac{\sqrt{2}}{4} & -A_2\frac{\sqrt{2}}{4} \\ 0 & -4A_3 & A_2\frac{\sqrt{2}}{4} & -A_2\frac{\sqrt{2}}{4} & -A_2\frac{\sqrt{2}}{4} & 4A_3 + \frac{\sqrt{2}}{4} \end{bmatrix}$$

Setting $A_1 = A_3 = A$ and $A_2 = A'$: Global stiffness matrix:

$$K = \frac{E}{4}\begin{bmatrix} 4A & 0 & -4A & 0 & 0 & 0 \\ 0 & 4A & 0 & 0 & 0 & -4A \\ -4A & 0 & 4A + A'\sqrt{2} & -A'\frac{\sqrt{2}}{4} & -A'\frac{\sqrt{2}}{4} & A'\frac{\sqrt{2}}{4} \\ 0 & 0 & -A'\frac{\sqrt{2}}{4} & A'\frac{\sqrt{2}}{4} & A'\frac{\sqrt{2}}{4} & -A'\frac{\sqrt{2}}{4} \\ 0 & 0 & -A'\frac{\sqrt{2}}{4} & A'\frac{\sqrt{2}}{4} & A'\frac{\sqrt{2}}{4} & -A'\frac{\sqrt{2}}{4} \\ 0 & -4A & A'\frac{\sqrt{2}}{4} & -A'\frac{\sqrt{2}}{4} & -A'\frac{\sqrt{2}}{4} & 4A + \frac{\sqrt{2}}{4} \end{bmatrix}$$

And comparing this matrix with the one we obtained from the triangular plate:

$$K^e = \frac{E}{4(1-v^2)}\begin{bmatrix} 1.5/A & 0.5/A & -1/A & -0.5/A & -0.5/A & 0 \\ 0.5/A & 1.5/A & 0 & -0.5/A & -0.5/A & -1/A \\ -1/A & 0 & 1/A & 0 & 0 & 0 \\ -0.5/A & -0.5/A & 0 & 0.5/A & 0.5/A & 0 \\ -0.5/A & -0.5/A & 0 & 0.5/A & 0.5/A & 0 \end{bmatrix}$$

2.2 Conclusion

Establishing a value of A' that allows us to resemble the stiffness matrix of a one-dimensional problem with the stiffness matrix of a plane stress problem makes no sense. This is because the spatial conditions and properties of the materials are different. Hooke's law for one-dimensional problems does not consider shear stress in the elements, so the relationship between the parallel and perpendicular deformations to the plane of application of forces (Poisson coefficient) does not come into play, while in a plane stress problem this will be reflected in the application of the constitutive matrix of the element.

2.3 Solution with $v \neq 0$

If we now consider the Poisson constant to be $v \neq 0$, the following stiffness matrix is obtained:

$$K^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \frac{E}{1-v^2}\begin{bmatrix} v & 1 & 0 \\ 1 & v & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Where:

$$A = \frac{1}{2}det(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}) = \frac{1}{2}$$

$$K^e = \frac{E}{4(1-v^2)} \begin{bmatrix} 3-v & 1+v & -2 & -1+v & -1+v & -2v \\ 1+v & 3-v & -2v & -1+v & -1+v & -2 \\ -2 & -2v & 2 & 0 & 0 & 2v \\ -1+v & -1+v & 0 & 1-v & 1-v & 0 \\ -1+v & -1+v & 0 & 1-v & 1-v & 0 \\ -2v & -2 & 2v & 0 & 0 & 2 \end{bmatrix}$$

2.4 Conclusion

The Poisson coefficient describes the expansion/contraction of material in a perpendicular direction to the applied force, therefore, it is now reflected in the stiffness matrix values that relate perpendicular axes such as the case of the stiffness presented in node 3 on the "y" axis in relation to the force that is applied in the direction of the "x" axis in node 1. due to the presence of shear stress. For the second case, in which we have a problem with one-dimensional elements, the Poisson coefficient does not play any roll, so it does not reflect any difference in the stiffness matrix, therefore it will remain the same .