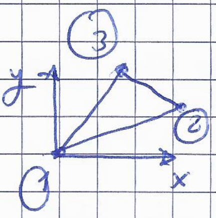
02/03/2020

Assignment 3.1

1) $E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}$ $\quad h = 1$



1 $(0,0)$

2 $(3,1)$

3 $(2,2)$

$A = \frac{1}{2} b \cdot a = 2$

$$K_{\pi}^{(e)} = \int_{\Omega^{(e)}} h B^T E B \, d\Omega^{(e)} = \frac{h}{4A} B^T E B =$$

$$= \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where:

$x_{12} = 0 - 3 = -3 \qquad y_{12} = 0 - 1 = -1$

$x_{13} = 0 - 2 = -2 \qquad y_{13} = 0 - 2 = -2$

$x_{21} = 3 - 0 = 3 \qquad y_{21} = 1 - 0 = 1$

$x_{31} = 2 - 0 = 2 \qquad y_{31} = 2 - 0 = 2$

$x_{32} = 2 - 3 = -1 \qquad y_{32} = 2 - 1 = 1$

$x_{23} = 3 - 2 = 1 \qquad y_{23} = 1 - 2 = -1$

$$K_{\pi} = \frac{1}{4 \cdot 2} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

$$K_{\pi} = \begin{bmatrix} 18{,}75 & 9{,}375 & -12{,}5 & -6{,}25 & -6{,}25 & -3{,}125 \\ 9{,}375 & 18{,}75 & 6{,}25 & 12{,}5 & -15{,}625 & -31{,}25 \\ -12{,}5 & 6{,}25 & 75 & -37{,}5 & -62{,}5 & 31{,}25 \\ -6{,}25 & 12{,}5 & -37{,}5 & 75 & 43{,}75 & -87{,}5 \\ -6{,}25 & -15{,}625 & -62{,}5 & 43{,}75 & 68{,}75 & -28{,}125 \\ -3{,}125 & -31{,}25 & 31{,}25 & -87{,}5 & -28{,}125 & 118{,}75 \end{bmatrix}$$

2) The sum of the rows (and columns) 1, 3, 5 of K^e as well as the sum of rows (and colums) 2, 4 and 6 vanish.

this is due to be the rows and columns of $K^{(e)}$ linear combinations of each other. we see there that there are unsuppressed rigid body motions. The triangle still "floats" in the xy plane. there is not applied any physical support condition such as displacement bounday conditions yet.

For the free-free structure, the complete force system is in self equilibrium. In the current case, the element stiffness equations are merged into the master stiffness equation.

Assignment 3.2

For plane stress, $K^{(e)} = \int_{\Omega^{(e)}} h B^T E B \, d\Omega^{(e)} = \frac{h}{4A} B^T E B$

a) As we stated before,

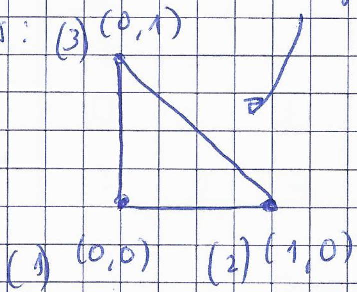
$$K_{tr}^{(e)} = \frac{h}{4A} B^T E B = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

(*) This can also be done in terms of the natural element coordinates.

since the elastic matrix for plane stress is:

$$\frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

and



$N_1 = 1 - \xi - \eta$

$N_2 = \xi$

$N_3 = \eta$

$y_{12} = 0 \qquad x_{12} = -1$

$y_{21} = 0 \qquad x_{21} = 1$

$y_{13} = -1 \qquad x_{13} = 0$

$y_{31} = 1 \qquad x_{31} = 0$

$y_{32} = 1 \qquad x_{32} = -1$

$y_{23} = -1 \qquad x_{23} = 1$

our system reads as:

$\frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \eta}{\partial x} = 0$

$\frac{\partial \xi}{\partial y} = 0 \qquad \frac{\partial \eta}{\partial y} = 1$

$\frac{\partial N_3}{\partial x} = 0 \qquad \frac{\partial N_3}{\partial y} = 1 \qquad \frac{\partial N_1}{\partial x} = -1 \qquad \frac{\partial N_1}{\partial y} = -1$

$\frac{\partial N_2}{\partial x} = 1 \qquad \frac{\partial N_2}{\partial y} = 0$

$$K_{tr}^{e} = \frac{hE}{4A(1-\nu^2)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K_{tr} = \frac{hE}{4A(1-\nu^2)} \begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

since $h=1$

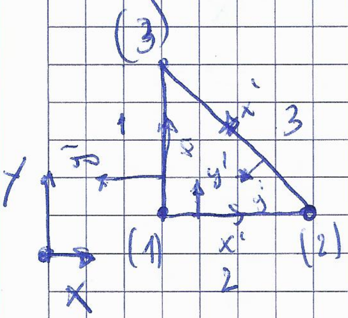
$a=1$

$A=\frac{1}{2}$

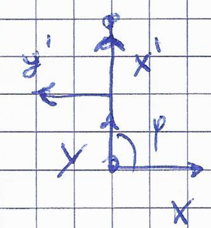
if $\nu=0$

$$\tilde{K}_{tr} = \frac{E}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & \cancel{0}2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

For the three-bars model, first localizing the element stiffness matrices



For the first element, 1

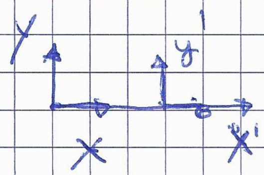


$c = \cos 90 = 0$

$s = \sin 90 = 1$

$$K^{(1)} = EA^{(1)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For the second element, 2

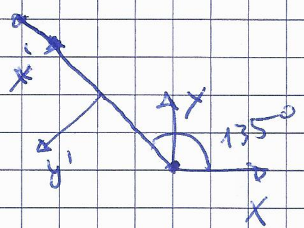


$c = \cos 0 = 1$

$s = \sin 0 = 0$

$$K^{(2)} = EA^{(2)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For the third element, 3



$l^{(3)} = \sqrt{a^2 + a^2} = \sqrt{2}$

$c = \cos 135 = -\frac{1}{\sqrt{2}}$

$s = \sin 135 = \frac{1}{\sqrt{2}}$

$$K^{(3)} = \frac{EA^{(3)}}{2\sqrt{2}} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

if we now globalize the element stiffness matrices,

$$K^{(1)} = EA^{(1)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K^{(2)} = EA^{(2)} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(3)} = \frac{EA^{(3)}}{2\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

if we now assemble the global stiffness matrix, $K_{bar} = K^{(1)} + K^{(2)} + K^{(3)}$

$$K_{bar} = E \begin{bmatrix} A^{(2)} & 0 & -A^{(2)} & 0 & 0 & 0 \\ 0 & A^{(1)} & 0 & 0 & 0 & -A^{(1)} \\ -A^{(2)} & 0 & A^{(2)} - \frac{A^{(3)}}{2\sqrt{2}} & \frac{-A^{(3)}}{2\sqrt{2}} & \frac{-A^{(3)}}{2\sqrt{2}} & \frac{-A^{(3)}}{2\sqrt{2}} \\ 0 & 0 & \frac{-A^{(3)}}{2\sqrt{2}} & \frac{A^{(3)}}{2\sqrt{2}} & \frac{A^{(3)}}{2\sqrt{2}} & -\frac{A^{(3)}}{2\sqrt{2}} \\ 0 & 0 & \frac{-A^{(3)}}{2\sqrt{2}} & \frac{A^{(3)}}{2\sqrt{2}} & \frac{A^{(3)}}{2\sqrt{2}} & -\frac{A^{(3)}}{2\sqrt{2}} \\ 0 & -A^{(1)} & \frac{A^{(3)}}{2\sqrt{2}} & \frac{-A^{(3)}}{2\sqrt{2}} & \frac{-A^{(3)}}{2\sqrt{2}} & A^{(1)} + \frac{A^{(3)}}{2\sqrt{2}} \end{bmatrix}$$

b) Finally, as we know that $A_1 = A_2$ and A_3, we let the next ~~for the~~ stated for the following exercise.

$A_1 = A_2 = A$

$A_3 = A'$

$$K_{bar} = E \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A - \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & -A & +\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & A + \frac{A'}{2\sqrt{2}} \end{bmatrix}$$

there is no set of values to make both stiffness matrices equivalent.

if we choose $A = \frac{3}{4}$, $A' = \frac{\sqrt{2}}{2}$ for K_{11}, K_{22}, K_{44}, K_{55},

we can make the stiffness matrices equal on the main diagonal.

$$K_{bar} = \frac{E}{4} \begin{bmatrix} 3 & 0 & -3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & -3 \\ -3 & 0 & 4 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & -3 & 1 & -1 & -1 & 4 \end{bmatrix}$$

$$K_{tr} = \frac{E}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

the reason why these two matrices are not equivalent comes from the fact that the bar only copes with axial tension and compression since it is considered as a plate (empty body). The turner triangle is considered as a continuum body (solid body) accounting for any applied force on the whole body (or displacement).
this fact, gives / brings more zero values on the bar stiffness matrix if it is compared with respect to the triangle stiffness matrix which is more dense.

d) Recalling the already obtained matrix K,

$$K = \frac{E}{2(1-\nu^2)} \begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

if poisson's ratio is taken into account, the stiffness values on the diagonal terms are reduced. The deformation generated by the presence of the Poisson efects acts in the same direction of the imposed ~~4th~~ or applied external forces.

Poisson's ratio also induces equilibrium in the system by increasing the off-diagonal terms of the stiffness matrix.