MASTER'S DEGREE NUMERICAL METHODS IN ENGINEERING

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Homework 3: Plane stress problem and linear triangle

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Assignment 3.1

1. Compute the entries of K_e for the following plane stress triangle:

$$x_1 = 0,\ y_1 = 0,\ x_2 = 3,\ y_2 = 1,\ x_3 = 2,\ y_3 = 2$$

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix},\ h = 1$$

Partial result: $K_{11} = 18.75$ and $K_{66} = 118.75$

2. Show that the sum of the rows (and columns) 1, 3 and 5 of K_e as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

Assignment 3.2

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are $E,\ \nu$. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

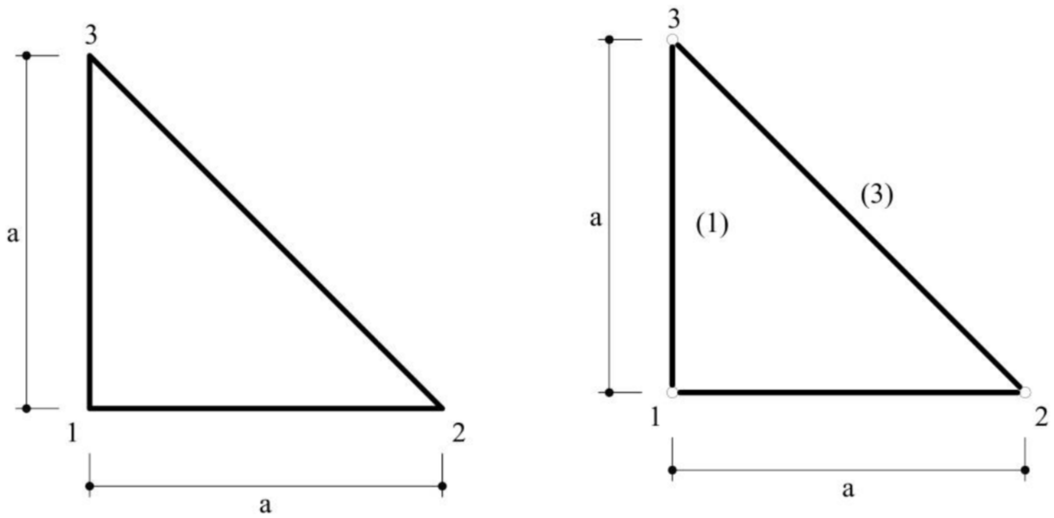

Figure 1: Triangular elements

(a) Calculate the stiffness matrix Ke for both models.

(b) Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

(c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

(d) Solve question (a) considering $\nu \neq 0$ and extract some conclusions.

Note: To solve this assignment it's recommended to check the features of the linear triangle in presentation "CSMD-05-Linear-Triangle". Some comments will be given in the next class.

1 Resolution

1.1 Assignment 3.1: First task

Compute the entries of K_e for the following plane stress triangle.

$$x_1 = 0,\ y_1 = 0,\ x_2 = 3,\ y_2 = 1,\ x_3 = 2,\ y_3 = 2$$

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix},\ h = 1$$

Partial result: $K_{11} = 18.75$ and $K_{66} = 118.75$

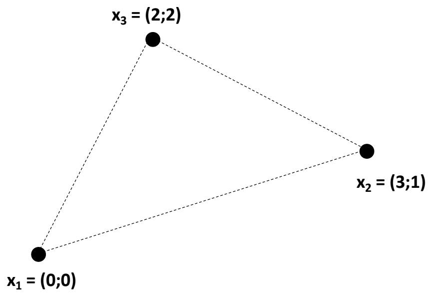

Figure 2: Triangle element.

The stiffness matrix of the triangle from Figure 2 is to be calculated. Departing from a stress-strain nodal displacement scheme, the stiffness matrix will be found using the following descriptions for displacement, strain and stress:

$$\delta\underline{u} = \underline{N_i}(x,y) \cdot \delta\underline{a}^{(e_i)}$$
$$\delta\underline{\varepsilon} = \underline{B_i}(x,y) \cdot \delta\underline{a}^{(e_i)}$$
$$\underline{\sigma} = \underline{D} \cdot \underline{B_i}(x,y) \cdot \underline{a}^{(e_i)}$$

Neglecting the body forces, the weak form of the stress-strain nodal displacement scheme is expressed as following:

$$\left[\delta\underline{a}^{(e)}\right]^T \cdot \left[\int\int_{A^e} \underline{B_i}^T(x,y) \cdot \underline{D} \cdot \underline{B_i}(x,y) \cdot \underline{a}^{(e_i)} \cdot t\ dA\right] = \left[\delta a^{(e)}\right]^T \cdot q^{(e)}$$

Which can be written as:

$$K^{(e)} \cdot a^{(e)} = q^{(e)}$$

The shape functions for N_i are calculated as:

$$N_i = \frac{1}{2A^{(e)}}(a_i + b_i x + c_i y)\ with\ i = 1,2,3$$

a_i b_i and c_i are as follows:

$$a_i = x_j\ y_k - x_k\ y_j$$
$$b_i = y_j - y_k$$
$$c_i = x_k - x_j$$

The $K_{ij}^{(e)}$ their terms are:

$$K_{ij}^{(e)} = \int\int_{A^e} B_i^T \cdot D \cdot B_j \cdot tdA = \int\int_{A^e} \frac{1}{2A^{(e)}} \cdot \begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix} \cdot \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \cdot \frac{1}{2A^{(e)}} \cdot \begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{bmatrix} \cdot t^{(e)}\ dA$$

For every element the terms of K_{ij} should be calculated and as the geometry of elements $1,\ 3,\ 4$ are equal, the will be part of a direct sum, for element 2 some difference will be encountered.

For elements $1,\ 3,\ 4$ the calculation of K is shown:

$$b_1 = y_2 - y_3 = -1.0$$
$$c_1 = x_3 - x_2 = -1.0$$
$$b_2 = y_3 - y_1 = 2.0$$
$$c_2 = x_1 - x_3 = -2.0$$
$$b_3 = y_1 - y_2 = -1.0$$
$$c_3 = x_2 - x_1 = 3.0$$

The thickness of the element is $t = h = 1$ and the area of triangle is calculated as:

$$2 \cdot A^{(e)} = det \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = 4$$

$$K^{(e)} = \int\int_{A^e} \frac{1}{2A} \begin{bmatrix} -1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & -1.0 \\ 2.0 & 0.0 & -2.0 \\ 0.0 & -2.0 & 2.0 \\ -1.0 & 0.0 & 3.0 \\ 0.0 & 3.0 & -1.0 \end{bmatrix} \cdot \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \cdot \frac{1}{2A} \begin{bmatrix} -1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & -1.0 \\ 2.0 & 0.0 & -2.0 \\ 0.0 & -2.0 & 2.0 \\ -1.0 & 0.0 & 3.0 \\ 0.0 & 3.0 & -1.0 \end{bmatrix}^T dA$$

$$K^{(e)} = \frac{1}{4A} \begin{bmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 600 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ -25 & -250 & 250 & -700 & -225 & 950 \end{bmatrix} = \begin{bmatrix} 18.75 & 9.38 & -12.50 & -6.25 & -6.25 & -3.13 \\ 9.38 & 18.75 & 6.25 & 12.50 & -15.63 & -31.25 \\ -12.50 & 6.25 & 75.00 & -37.50 & -62.50 & 31.25 \\ -6.25 & 12.50 & -37.50 & 75.00 & 43.75 & -87.50 \\ -6.25 & -15.63 & -62.50 & 43.75 & 68.75 & -28.13 \\ -3.13 & -31.25 & 31.25 & -87.50 & -28.13 & 118.75 \end{bmatrix}$$

1.2 Assignment 3.1: Second task

Show that the sum of the rows (and columns) 1, 3 and 5 of K_e as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

The columns 1, 3 and 5 multiply the $u_{x1},\ u_{x2}$ and u_{x3} coordinates respectively. They must sum an overall of zero because, as the stiffness matrix relates displacements with stresses (through strains), one possible displacement is a rigid-body displacement which do not generate any strains nor stress. Therefore, a rigid-body displacement scheme has the same value of displacement for every coordinate for example $[u_{x1},\ u_{x2},\ u_{x3}] = [1,\ 1,\ 1]$ and this configuration will register a displacement, without any strain, then no stresses and the rigid-body displacement can be represented. The same happens on columns 2, 4 and 6 for coordinates $u_{y1},\ u_{y2}$ and u_{y3}. For the rows, the symmetry of the matrix makes the rows to be equal too.

1.3 Assignment 3.2

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are $E,\ \nu$. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.

(a) Calculate the stiffness matrix Ke for both models.

(b) Is there any set of values for cross sections $A_1 = A_2 = A$ and $A_3 = A'$ to make both stiffness matrix equivalent: $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

(c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

(d) Solve question (a) considering $\nu \neq 0$ and extract some conclusions.

Turner: This method is the applied in the previous exercise, therefore the results will be plotted with less detail.

$$2 \cdot A^{(e)} = det \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = 1$$

$$K^{(e)} = \int\int_{A^e} \frac{1}{2A} \begin{bmatrix} -1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & -1.0 \\ 1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & 1.0 \\ -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -1.0 \end{bmatrix} \cdot \begin{bmatrix} E & \nu E & 0 \\ \nu E & E & 0 \\ 0 & 0 & \frac{E}{2\cdot(1+\nu)} \end{bmatrix} \cdot \frac{1}{2A} \begin{bmatrix} -1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & -1.0 \\ 1.0 & 0.0 & -1.0 \\ 0.0 & -1.0 & 1.0 \\ -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -1.0 \end{bmatrix}^T dA$$

$$K^{(e)} = \frac{1}{4A} \cdot \begin{bmatrix} E+G & \nu E + G & -E+G & \nu E - G & E & G \\ & E+G & -\nu E + G & E - G & \nu E & G \\ & & E+G & -\nu E - G & -E & G \\ & & & E+G & \nu E & -G \\ & sym. & & & E & 0 \\ & & & & & G \end{bmatrix}$$

In the first task for this exercise (task (a)), asks for the result if $\nu = 0$, then $G = E/2$:

$$K^{(e)} = \frac{E}{2} \begin{bmatrix} 3/2 & 1/2 & -1/2 & -1/2 & 1 & 1/2 \\ & 3/2 & 1/2 & 1/2 & 0 & 1/2 \\ & & 3/2 & -1/2 & -1 & 1/2 \\ & & & 3/2 & 0 & -1/2 \\ & sym. & & & 1 & 0 \\ & & & & & 1/2 \end{bmatrix}$$

Bar element: To calculate the stiffness matrix of a bar element, the theory of *the direct stiffness method* from the first class is used:

$$K^{(e)} = \frac{E^{(e)} \cdot A^{(e)}}{L^{(e)}} \cdot \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

where $s,\ c$ are $sin(\theta_i),\ cos(\theta_i)$ respectively. The angle of each of the bars are:

$$\theta_1 = 90,\ \theta_2 = 0,\ \theta_3 = 315$$

$$K^{(1)} \cdot \underline{u}^{(1)} = \frac{E \cdot A}{1} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{x1} \\ u_{y1} \end{bmatrix}$$

$$K^{(2)} \cdot \underline{u}^{(2)} = \frac{E \cdot A}{1} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x1} \\ u_{y1} \end{bmatrix}$$

$$K^{(3)} \cdot \underline{u}^{(3)} = \frac{E \cdot A_3}{1} \cdot \begin{bmatrix} 0.354 & -0.354 & -0.354 & 0.354 \\ -0.354 & 0.354 & 0.354 & -0.354 \\ -0.354 & 0.354 & 0.354 & -0.354 \\ 0.354 & -0.354 & -0.354 & 0.354 \end{bmatrix} \cdot \begin{bmatrix} u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Then the global stiffness matrix is calculated after assembling the three elemental matrices:

$$K_G \cdot \underline{u} = E \cdot \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A + 0.354A_3 & -0.354A_3 & -0.354A_3 & 0.354A_3 \\ & & & 0.354A_3 & 0.354A_3 & -0.354A_3 \\ & sym. & & & 0.354A_3 & -0.354A_3 \\ & & & & & A + 0.354A_3 \end{bmatrix} \cdot \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Task *(b)* asks to find values of A and A_3 to make both stiffness matrices as similar as possible:

$$\begin{bmatrix} 3/4 & 1/4 & -1/4 & -1/4 & 1/2 & 1/4 \\ & 3/4 & 1/4 & 1/4 & 0 & 1/4 \\ & & 3/4 & -1/4 & -1/2 & 1/4 \\ & & & 3/4 & 0 & -1/4 \\ & sym. & & & 1/2 & 0 \\ & & & & & 1/4 \end{bmatrix} = \begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ & A & 0 & 0 & 0 & -A \\ & & A + 0.5/\sqrt{2}A_3 & -0.5/\sqrt{2}A_3 & -0.5/\sqrt{2}A_3 & 0.5/\sqrt{2}A_3 \\ & & & 0.5/\sqrt{2}A_3 & 0.5/\sqrt{2}A_3 & -0.5/\sqrt{2}A_3 \\ & sym. & & & 0.5/\sqrt{2}A_3 & -0.5/\sqrt{2}A_3 \\ & & & & & A + 0.5/\sqrt{2}A_3 \end{bmatrix}$$

Clearly there are no values of A and A_3 to make both matrices equal. One of the most similar stiffness matrices will be found if the diagonal terms are similar:

$$A = \frac{3}{4},\ A_3 = \frac{1}{\sqrt{2}}$$

Task *(c)* asks to explain why these two matrices are not equal.
The reason is that Turner formulation is made for a 2D element, while the direct stiffness method is formulated for a 1D bar with nothing more than axial stiffness, therefore, the elements will behave completely different.

Task *(d)* asks to solve the problem with $\nu \neq 0$.

$$K^{(e)} = \frac{E}{2} \cdot \begin{bmatrix} 1 + \frac{1}{2\cdot(1+\nu)} & \nu + \frac{1}{2\cdot(1+\nu)} & -1 + \frac{1}{2\cdot(1+\nu)} & \nu - \frac{1}{2\cdot(1+\nu)} & 1 & \frac{1}{2\cdot(1+\nu)} \\ & 1 + \frac{1}{2\cdot(1+\nu)} & -\nu + \frac{1}{2\cdot(1+\nu)} & 1 - \frac{1}{2\cdot(1+\nu)} & \nu & \frac{1}{2\cdot(1+\nu)} \\ & & 1 + \frac{1}{2\cdot(1+\nu)} & -\nu - \frac{1}{2\cdot(1+\nu)} & -1 & \frac{1}{2\cdot(1+\nu)} \\ & & & 1 + \frac{1}{2\cdot(1+\nu)} & \nu & -\frac{1}{2\cdot(1+\nu)} \\ & sym. & & & 1 & 0 \\ & & & & & \frac{1}{2\cdot(1+\nu)} \end{bmatrix}$$

The main difference between the $\nu = 0$ and the $\nu \neq 0$ is seen in some coordinates where it used to be a zero, now there is not. Also, the stiffness over the main diagonal is lower for the $\nu \neq 0$ case as the $1/2(1+\nu)$ term are now smaller.

2 Conclusions

Two main tasks were solved in the frame of *Plane stress problem and linear triangle*. In the first, the stiffness matrix of a triangular element was obtained. Afterwards another triangle was analysed using both 2D description and a 1D bar description and compared. As a result, the matrices shown major differences due to the physics being described by each of the models.