Universitat Politecnica de Catalunya CIMNE
Master in Computational Mechanics

Computational Structural Mechanics & Dynamics

Assignment 3

Plane Stress Problem & Linear Triangle

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Assignment 3.1 :

1. Compute the entries of K_e for the following plane stress triangle:

$$x_1 = 0,\ y_1 = 0,\ x_2 = 3,\ y_2 = 1,\ x_3 = 2,\ y_3 = 2$$

$$\boldsymbol{E} = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix},\quad h = 1$$

Partial result: $K_{11} = 18{,}75$, $K_{66} = 118{,}75$.

2. Show that the sum of the rows (and columns) 1, 3 and 5 of $\mathbf{K}^e$ as well as the sum of rows (and columns) 2, 4 and 6 must vanish, and explain why.

Solution:

1)

The stiffness K^e for the given plane stress triangle is given by:

$$K^e = \int hB^T EB\, d\Omega$$

$$\therefore\ K^e = B^T EB \int h\, d\Omega$$

The matrices B and E are given by:

$$B = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$E = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Since, B and E are constant over entire triangle area, and the value of h is given to be 1, we have,

$$K^e = B^T EB * A$$

$$K^e = \frac{A}{4A^2}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{4A}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

Now, the area of triangle is given by,

$$\mathrm{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$\mathrm{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\mathrm{A} = 2$$

Substituting the value of area in the stiffness formula above, we get,

$$K^e = \frac{1}{8}\begin{bmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 600 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ -25 & -250 & 250 & -700 & -225 & 950 \end{bmatrix}$$

$$K^e = \begin{bmatrix} 18.75 & 9.38 & -12.50 & -6.25 & -6.25 & -3.13 \\ 9.38 & 18.75 & 6.25 & 12.50 & -15.63 & -31.25 \\ -12.50 & 6.25 & 75 & -37.50 & -62.50 & 31.25 \\ -6.25 & 12.50 & -37.50 & 75 & 43.75 & -87.50 \\ -6.25 & -15.63 & -62.50 & 43.75 & 68.75 & -28.13 \\ -3.13 & -31.25 & 31.25 & -87.50 & -28.13 & 118.75 \end{bmatrix}$$

2)

From the above stiffness matrix we observe that, the sum of columns (1+3+5) and (2+4+6) is zero. The same applies for rows as well.

Columns (1,3,5): $$\begin{bmatrix} 18.75 \\ 9.38 \\ -12.50 \\ -6.25 \\ -6.25 \\ -3.13 \end{bmatrix} + \begin{bmatrix} -12.50 \\ 6.25 \\ 75 \\ -37.50 \\ -62.50 \\ 31.25 \end{bmatrix} + \begin{bmatrix} -6.25 \\ -15.63 \\ -62.50 \\ 43.75 \\ 68.75 \\ -28.13 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Columns (2,4,6): $$\begin{bmatrix} 9.38 \\ 18.75 \\ 6.25 \\ 12.50 \\ -15.63 \\ -31.25 \end{bmatrix} + \begin{bmatrix} -6.25 \\ 12.50 \\ -37.50 \\ 75 \\ 43.75 \\ -87.50 \end{bmatrix} + \begin{bmatrix} -3.13 \\ -31.25 \\ 31.25 \\ -87.50 \\ -28.13 \\ 118.75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

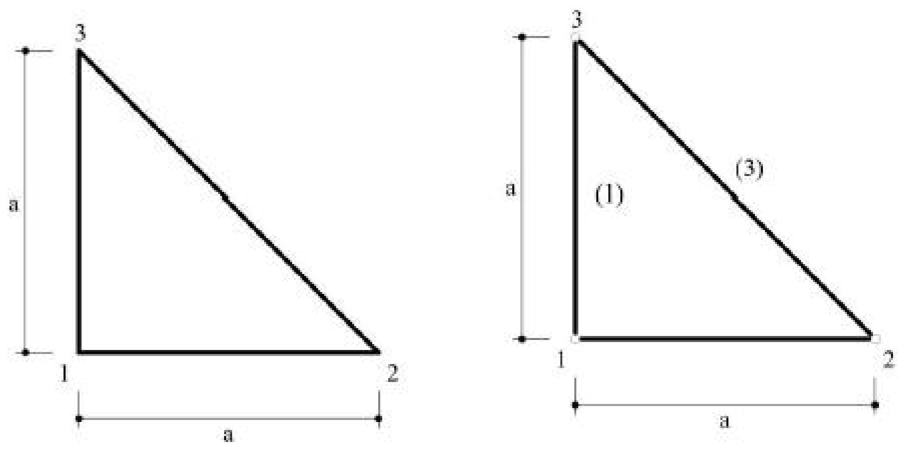
Same operation can be observed for the rows as well.

The reason for this is, in order to maintain equilibrium of the forces and reactions, the sum of alternate rows and columns is zero. Summation of alternate rows and columns tending to zero indicates a perfect balanced matrix. If the summation is not zero, then there would be no equilibrium in the forces and reactions acting on the structure which would result in giving misleading values of forces or displacements.

Assignment 3.2 :

Consider a plane triangular domain of thickness h, with horizontal and vertical edges have length a. Let's consider for simplicity $a = h = 1$. The material parameters are E,ν. Initially ν is set to zero. Two structural models are considered for this problem as depicted in the figure:

- A plane linear Turner triangle with the same dimensions.
- A set of three bar elements placed over the edges of the triangular domain. The cross sections for the bars are $A_1 = A_2$ and A_3.



a) Calculate the stiffness matrix K^e for both models.

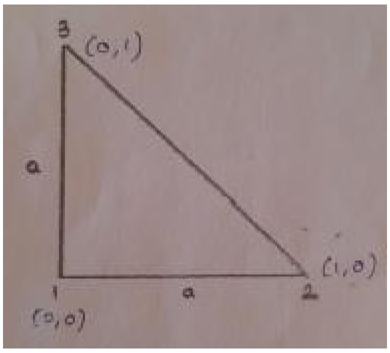
b) Is there any set of values for cross sections A_1=A_2=A and A_3=A' to make both stiffness matrix equivalent: $K_{bar} = K_{triangle}$? If not, which are these values to make them as similar as possible?

c) Why these two stiffness matrix are not equivalent? Find a physical explanation.

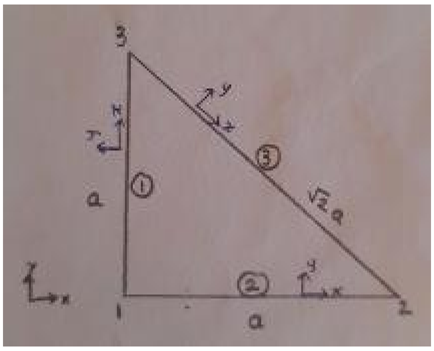
d) Solve question a) considering ν≠0 and extract some conclusions.

Solution:

a)



Turner Triangle



Bar Elements

a = h = 1, ν = 0

$x_1 = 0, x_2 = 1, x_3 = 0; y_1 = 0, y_2 = 0, y_3 = 1$

- K^e for the plane linear turner triangle-

$$K_{triangle} = \int hB^T EB \, d\Omega$$

$$K_{triangle} = \frac{A}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Here, ν = 0 so **E** matrix becomes,

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The area of triangle is given by,

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \frac{1}{2}$$

The stiffness matrix becomes,

$$K_{triangle} = \frac{1}{4*0.5} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K_{triangle} = \frac{E}{2}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K_{triangle} = E\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

- K^e for the bar elements-

$E^1 = E^2 = E^3 = E$
$A_1 = A_2$
$L^1 = L^2 = 1; L^3 = \sqrt{2}$
Element 1: $\alpha_1 = \frac{\pi}{2}$, Element 2: $\alpha_2 = 0$, Element 3: $\alpha_3 = \frac{3\pi}{4}$

$$K^e = \frac{E^e A^e}{L^e}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$K^1 = \frac{EA_1}{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad K^2 = \frac{EA_1}{1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^3 = \frac{EA_3}{\sqrt{2}}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

Augmenting the matrices,

$$K^1 = E\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_1 & 0 & 0 & 0 & A_1 \end{bmatrix} \qquad K^2 = E\begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A_1 & 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^3 = E\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \end{bmatrix}$$

The global stiffness matrix for the bar elements is,

$$K_{bar} = E\begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_1 & 0 & A_1 + \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & 0 & -\frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} \\ 0 & -A_1 & \frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & -\frac{A_3}{2\sqrt{2}} & A_1 + \frac{A_3}{2\sqrt{2}} \end{bmatrix}$$

b)

For $A_1 = A_2 = A$ and $A_3 = A'$ the K_{bar} and $K_{triangle}$ cannot be made exactly the same because of uneven distribution of the matrix elements. In the two matrices, some places where there are zeros in one matrix, there is some other non-zero value in the other matrix which cannot be made zero. Hence, both the matrices cannot be made exactly the same. Though, some elements can be made similar. For example, have a look at both the matrices below,

$$K_{triangle} = E\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$K_{bar} = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & -A \\ -A & 0 & A+\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & 0 & -\frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} \\ 0 & -A & \frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & -\frac{A'}{2\sqrt{2}} & A+\frac{A'}{2\sqrt{2}} \end{bmatrix}$$

The diagonal elements can be equated to get the values which can make them little bit similar. So, A can take the value as A = $\frac{3}{4}$ and A' should take the value as A' = $\frac{1}{\sqrt{2}}$.

c)

The stiffness matrices K_{bar} and $K_{triangle}$ are different on account of different elements used. Though the shape looks similar but the material distribution in the triangular element and the bar element is different.

The triangular element covers the entire triangular area within its boundaries and the stress and stiffness is distributed over the entire area of the triangular element. Whereas, in the bar elements formed like a triangle, the area covered is only along the axial direction of the bar, i.e only around the boundaries of the 3 bars which form the triangle. The area in between the bars does not have any stiffness value.

Hence, a single triangular element would be more resistant to forces than the bar elements which form a triangle shape and it does not make sense in comparing their stiffness because of the difference in the properties.

d)

Considering ν ≠ 0, it can be observed that most of the terms in the matrix are affected by the value of ν.

Implementing this condition for the case of $K_{triangle,}$ the matrix equation becomes,

$$K_{triangle} = \frac{1}{4*0.5}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$K_{triangle} = \frac{E}{2(1-\nu^2)}\begin{bmatrix} \frac{3-\nu}{2} & \frac{1+\nu}{2} & -1 & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -\nu \\ \frac{1+\nu}{2} & \frac{3-\nu}{2} & -\nu & \frac{\nu-1}{2} & \frac{\nu-1}{2} & -1 \\ -1 & -\nu & 1 & 0 & 0 & \nu \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \frac{\nu-1}{2} & \frac{\nu-1}{2} & 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ -\nu & -1 & \nu & 0 & 0 & 1 \end{bmatrix}$$

The Poisson effect is considered in this matrix which means that shear effects will also be considered in stress. Considering the Poisson effect makes the element more resistant to applied forces, in terms to stiffness. The stiffness matrix with ν = 0 will certainly tend to induce more displacements, as it will offer less resistance to forces as compared to stiffness matrix with ν ≠ 0 which would be more resistant.

Discussions:

- In the first task, stiffness of the plane stress triangle was calculated and it was observed that the sum of alternate rows and columns is zero. This was necessary in order to prove the equilibrium of the forces and reactions acting on the structure being analysed.
- In the second task, stiffness was calculated for a triangular element and bar elements representing a triangle, and the values were compared. It was seen that, since the triangular element covers the entire area within the triangular domain as compared to the bar elements, therefore the stiffness value by triangular element would be more than the stiffness value by the bar elements. More resistance is offered by the triangular element. Displacements would be therefore more, in case of the bar elements as compared to the triangular element.
- Incorporating a non-zero value for the Poisson ratio counts for the shear stress effects as well, thus increasing the resistance as compared to a zero Poisson ratio.