Master on Numerical Methods in Engineering

Computational Structural Mechanics and Dynamics

Assignment 3

Plane stress problem and linear triangle

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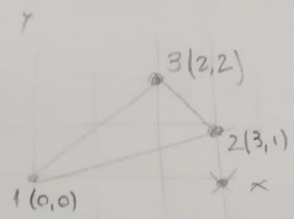
ASSIGNMENT ③ Plane Stress problem and linear Triangle

Assignment (3.1)

1). Compute the entries of K_e for the following plane stress triangle



Given data:

$$E = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \; ; \quad h = 1.$$

The element stiffness matrix is, in general, given by $K^e = \int_{\Omega^e} h \, B^T E B \, d\Omega$ where Ω^e is the triangle domain and h is the plate thickness.

Considering that for this exercise h is uniform and B and E are constants, Ke will be defined by:

$$K_e = A h \cdot \underbrace{B^T E B}_{\text{given data}}$$

In order to define Ke, B (strain-displacement matrix) must be computed firstly.

$$B = \frac{1}{2A} \left[\begin{array}{cc|cc|cc} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{array} \right] = \frac{1}{2A} \left[\begin{array}{cc|cc|cc} -1 & 0 & +2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{array} \right]$$

where $\begin{cases} x_{jk} = x_j - x_k \\ y_{jk} = y_j - y_k \end{cases}$

and $2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix} = 4 \; ; \; A = 2$

Resulting stiffness matrix for the element is:

$$\boxed{K^e = \frac{1}{8} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -1 & -2 & 2 & 3 & -1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 150 & 75 & -100 & -50 & -50 & -25 \\ 75 & 150 & 50 & 100 & -125 & -250 \\ -100 & 50 & 600 & -300 & -500 & 250 \\ -50 & 100 & -300 & 600 & 350 & -700 \\ -50 & -125 & -500 & 350 & 550 & -225 \\ -25 & -250 & 250 & 700 & -225 & 950 \end{bmatrix}}$$

(1)

(MOC)

2) Show that the sum of rows (and columns) 1, 3, and 5 of K^e as well as the sum of row (and column) 2, 4 and 6 must vanish. Explain why

→ Sum of 1,3,5 rows (and columns) of K^e:

$$\frac{1}{8}\begin{bmatrix}150 & 75 & -100 & -50 & -50 & -25\end{bmatrix} + \frac{1}{8}\begin{bmatrix}-100 & 50 & 600 & -300 & -500 & 250\end{bmatrix} + \frac{1}{8}\begin{bmatrix}-50 & -125 & -500 & 350 & 550 & -225\end{bmatrix} =$$

$$= \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

→ Sum of 2,4,6 rows (and columns) of K^e:

$$\frac{1}{8}\begin{bmatrix}75 & 150 & 50 & 100 & -175 & -250\end{bmatrix} + \frac{1}{8}\begin{bmatrix}-50 & 100 & -300 & 600 & 350 & -700\end{bmatrix} + \frac{1}{8}\begin{bmatrix}-25 & -250 & 250 & 700 & -225 & 950\end{bmatrix} =$$

$$= \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Sum equals zero guarantees equilibrium between internal and external forces.

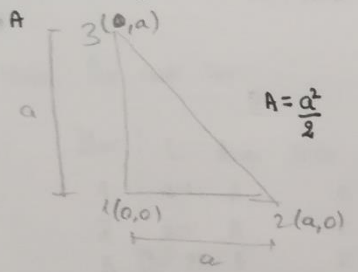
(2)

(MOC

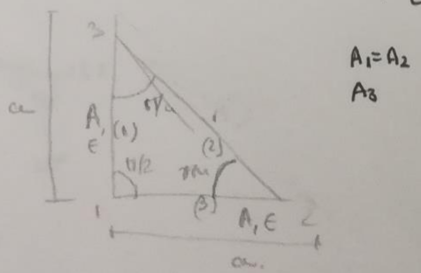
Assignment 3.2

Plane triangular domain of thickness h. Horizontal & vertical edges have length a.
For simplicity $a = h = 1$.
Material parameters E and ν; initially ν is set to zero. $(\nu = 0)$
Two structural models are considered for this problem:

- Plane linear triangle with same dimensions.



- Set of three bar elements placed over the edges of triangular domain.
  Cross sections: $A_1 = A_2$ and A_3. [MODEL B]



a) Calculate the stiffness matrix K^e for both models.

MODEL A / PLANE LINEAR TRIANGLE (TURNER)

Repeating same procedure as in Assignment 3.1; replacing values and considering that B and E are constant; these matrices have to be computed:

$$B = \frac{1}{2A}\left[\begin{array}{cc|cc|cc} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{array}\right] = \frac{1}{\cancel{2}\cdot\frac{a^2}{\cancel{2}}}\left[\begin{array}{cc|cc|cc} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{array}\right] = \frac{1}{\cancel{a^2}\,1}\left[\begin{array}{cc|cc|cc} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{array}\right].$$

Strain-displacement matrix

$$E = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \overset{\nu=0}{=} E\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Elastic constitutive matrix

Element stiffness matrix for a Turner triangle: $K^e = \frac{h}{4A} B^T \cdot E \cdot B = \frac{B^T \cdot E \cdot B}{2}$

$$K^e = \frac{E}{2}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \left[\begin{array}{cc|cc|cc} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{array}\right] =$$

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MODEL B | BAR ELEMENT – TRIANGULAR DOMAIN (3 bar elements)

Element stiffness matrix for this case is:

$$K^e = \frac{EA_e}{L^e}\begin{bmatrix} c^2 & sc & -c^2 & -cs \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & cs \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

where $\begin{cases} c \equiv \cos\alpha \\ s \equiv \sin\alpha \end{cases}$

Given data for each bar:

Bar	L	Area	Elastic module	Angle w.r.t x
1	$a=1$	A	E	90°
2	$a=1$	A	E	0
3	$\sqrt{2a^2}=2$	A'	E	45°

Computation of the K^e for each bar element:

- Element (1): $K^{(1)} = EA\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

- Element (2) $K^{(2)} = EA\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- Element (3) $K^{(3)} = \frac{EA'}{2\sqrt{2}}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

The global stiffness matrix iss assembled as follow:

$$K = E\begin{bmatrix} A & 0 & -A & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & 0 \\ -A & 0 & A+\sqrt{2}A'/4 & -\sqrt{2}A'/4 & -\sqrt{2}A'/4 & \sqrt{2}A'/4 \\ 0 & 0 & -\sqrt{2}A'/4 & \sqrt{2}A'/4 & \sqrt{2}A'/4 & -\sqrt{2}A'/4 \\ 0 & 0 & -\sqrt{2}A'/4 & \sqrt{2}A'/4 & \sqrt{2}A'/4 & -\sqrt{2}A'/4 \\ 0 & 0 & \sqrt{2}A'/4 & -\sqrt{2}A'/4 & -\sqrt{2}A'/4 & A+\sqrt{2}A'/4 \end{bmatrix}$$

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