Assignment 3.1

STUDENT:
Marcello Rubino

PAGE 1

1)
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

$$\begin{cases} E\nu = (1+\nu)(1-2\nu)\lambda \\ E = 2(1+\nu)\mu \end{cases}$$

$$\begin{cases} 2\nu\cancel{(1+\nu)}\mu = \cancel{(1+\nu)}(1-2\nu)\lambda \\ \leftarrow \end{cases} \qquad \begin{cases} 2\nu\mu = \lambda - 2\nu\lambda \\ \leftarrow \end{cases}$$

$$\begin{cases} 2\nu(\mu+\lambda) = \lambda \\ \leftarrow \end{cases} \qquad \begin{cases} \nu = \dfrac{\lambda}{2(\mu+\lambda)} \\ E = 2\left(1 + \dfrac{\lambda}{2(\mu+\lambda)}\right)\mu \end{cases} \qquad \begin{cases} \leftarrow \\ E = \cancel{2}\cdot\left(\dfrac{2(\mu+\lambda)+\lambda}{\cancel{2}(\mu+\lambda)}\right)\mu \end{cases}$$

$$E = \frac{(2\mu+3\lambda)\mu}{\mu+\lambda} \qquad \nu = \frac{\lambda}{2(\mu+\lambda)}$$

2)
- PLANE STRESS $\underline{E}$ MATRIX:

$$\underline{E} = \frac{E}{1-\nu^2}\begin{array}{c} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \end{array}\begin{array}{l} xx \\ yy \\ xy \end{array}$$

$$\rightarrow \underline{E} = \frac{4\mu(2\mu^2+5\lambda\mu+3\lambda^2)}{4\mu^2+8\lambda\mu+3\lambda^2}\cdot\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2\mu+\lambda}{4(\mu+\lambda)} \end{bmatrix}$$

- PLANE STRAIN $\underline{E}$ MATRIX

$$\underline{E} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\hookrightarrow \underline{E} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} =$$

3)

$$\underline{E}_\mu + \underline{E}_\lambda = \underbrace{\begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}}_{\underline{E}_\mu} + \underbrace{\begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{E}_\lambda}$$

4)

$$\underline{E}_\mu = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} \frac{E}{1+\nu} & 0 & 0 \\ 0 & \frac{E}{1+\nu} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$

$$\underline{E}_\lambda = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Assignment 3.2

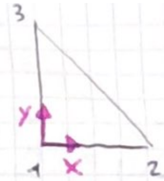
We consider the plane stress case, and initially $\nu = \emptyset$.

1) - (a)

$$\underline{E} = \begin{bmatrix} E & \emptyset & \emptyset \\ \emptyset & E & \emptyset \\ \emptyset & \emptyset & E/2 \end{bmatrix} \qquad \underline{B} = \frac{1}{2A} \begin{bmatrix} Y_{23} & 0 & Y_{31} & 0 & Y_{12} & 0 \\ 0 & X_{32} & 0 & X_{13} & 0 & X_{21} \\ X_{32} & Y_{23} & X_{13} & Y_{31} & X_{21} & Y_{12} \end{bmatrix}$$

where: $Y_{ij} = Y_i - Y_j$

$X_{ij} = X_i - X_j$



in this case :

$$\underline{B} = \frac{1}{2A} \cdot \begin{bmatrix} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{bmatrix}$$

h = costant, so:

$$\underline{K}^e_{tri} = A h\, \underline{B}^T \underline{E}\, \underline{B} = \frac{h}{4A} \begin{bmatrix} -a & 0 & -a \\ 0 & -a & -a \\ a & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix} \begin{bmatrix} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{bmatrix}$$

$$= \frac{h}{4A} \begin{bmatrix} -aE & 0 & -aE/2 \\ 0 & -aE & -aE/2 \\ aE & 0 & 0 \\ 0 & 0 & aE/2 \\ 0 & 0 & aE/2 \\ 0 & aE & 0 \end{bmatrix} \begin{bmatrix} & & \\ & " & \\ & & \end{bmatrix} =$$

$$\underline{K}^e_{tri} = \frac{h}{2\cancel{a^2}} \begin{bmatrix} \frac{3\cancel{a^2}E}{2} & \frac{\cancel{a^2}E}{2} & -\cancel{a^2}E & -\frac{\cancel{a^2}E}{2} & -\frac{\cancel{a^2}E}{2} & 0 \\ \frac{\cancel{a^2}E}{2} & \frac{3\cancel{a^2}E}{2} & 0 & -\frac{\cancel{a^2}E}{2} & -\frac{\cancel{a^2}E}{2} & -\cancel{a^2}E \\ -\cancel{a^2}E & 0 & \cancel{a^2}E & 0 & 0 & 0 \\ -\frac{\cancel{a^2}E}{2} & -\frac{\cancel{a^2}E}{2} & 0 & \frac{\cancel{a^2}E}{2} & \frac{\cancel{a^2}E}{2} & 0 \\ -\frac{\cancel{a^2}E}{2} & -\frac{\cancel{a^2}E}{2} & 0 & \frac{\cancel{a^2}E}{2} & \frac{\cancel{a^2}E}{2} & 0 \\ 0 & -\cancel{a^2}E & 0 & 0 & 0 & \cancel{a^2}E \end{bmatrix} = \begin{bmatrix} \frac{3Eh}{4} & \frac{Eh}{4} & -\frac{Eh}{2} & -\frac{Eh}{4} & -\frac{Eh}{4} & 0 \\ \frac{Eh}{4} & \frac{3Eh}{4} & 0 & -\frac{Eh}{4} & -\frac{Eh}{4} & \frac{Eh}{2} \\ -\frac{Eh}{2} & 0 & \frac{Eh}{2} & 0 & 0 & 0 \\ -\frac{Eh}{4} & -\frac{Eh}{4} & 0 & \frac{Eh}{4} & \frac{Eh}{4} & 0 \\ -\frac{Eh}{4} & -\frac{Eh}{4} & 0 & \frac{Eh}{4} & \frac{Eh}{4} & 0 \\ 0 & -\frac{Eh}{2} & 0 & 0 & 0 & \frac{Eh}{2} \end{bmatrix}$$

1) -(b)

$$\underset{3-1}{\tilde{\underline{K}}^e_{(1)}} = \begin{bmatrix} \frac{EA_1}{a} & 0 & -\frac{EA_1}{a} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA_1}{a} & 0 & \frac{EA_1}{a} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \bar{\underline{K}}^e_{(1)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{EA_1}{a} & 0 & -\frac{EA_1}{a} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{EA_1}{a} & 0 & \frac{EA_1}{a} \end{bmatrix}$$

$$\tilde{\underline{K}}^e_{(2)} = \bar{\underline{K}}^e_{(2)} = \begin{bmatrix} \frac{EA_2}{a} & 0 & -\frac{EA_2}{a} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA_2}{a} & 0 & \frac{EA_2}{a} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1-2

$$\tilde{\underline{K}}^e_{(3)} = \begin{bmatrix} \frac{EA_3}{\sqrt{2}a} & 0 & -\frac{EA_3}{\sqrt{2}a} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA_3}{\sqrt{2}a} & 0 & \frac{EA_3}{\sqrt{2}\,a} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \bar{\underline{K}}^e_{(3)} = \begin{bmatrix} \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} \\ -\frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} \\ -\frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} \\ \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & \frac{EA_2\sqrt{2}}{4a} \end{bmatrix}$$

2-3

↓ Knowing that $A_1 = A_2 = A$

$$\underline{K}^e_{bar} = \begin{bmatrix} \frac{EA}{a} & 0 & -\frac{EA}{a} & 0 & 0 & 0 \\ 0 & \frac{EA}{a} & 0 & 0 & 0 & -\frac{EA}{a} \\ -\frac{EA}{a} & 0 & \frac{EA}{a} + \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} \\ 0 & 0 & -\frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} \\ 0 & 0 & -\frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} \\ 0 & -\frac{EA}{a} & \frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & -\frac{EA_3\sqrt{2}}{4a} & \frac{EA}{a} + \frac{EA_3\sqrt{2}}{4a} \end{bmatrix}$$

2) Reading the two matrices we can see that there is no possible set of values for
3) $A_1 = A_2 = A$ and A_3 to make them equivalent. We can only see similarities in the elements 1-1, 2-2, only if $\boxed{\frac{A}{a} = \frac{3h}{4}}$; in elements 3-3 and 6-6 or in 4-4, 4-5, 5-4, 5-5, where there is equivalence if $\boxed{\frac{A_3\sqrt{2}}{4a} = \frac{h}{4}}$. The rest of the matrices are completely different, because for some dof the bars structure doesn't brings any internal force while the triangle does. ($\underline{K}_{e_{tri}}$ is more "powerful", meaning that the possible internal power is much greater.) - If the solution is linear than it's really difficult or impossible to use triangular structures of bars because they don't bring any contribution in some degrees of freedom.

4) Considering nowidering $\nu \neq 0$, we have that:

$$\underline{K}^{e}_{tri,\nu} = Ah \cdot \underline{B}^T \underline{E}\, \underline{B} = \frac{h}{4A} \begin{bmatrix} -a & 0 & -a \\ 0 & -a & -a \\ a & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \frac{E}{1-\nu^2} \cdot \underline{B}$$

$$= \frac{Eh}{2a^2(1-\nu^2)} \begin{bmatrix} -a & -a\nu & -a\frac{(1-\nu)}{2} \\ -a\nu & -a & -a\frac{(1-\nu)}{2} \\ a & a\nu & 0 \\ 0 & 0 & a\frac{(1-\nu)}{2} \\ 0 & 0 & a\frac{(1-\nu)}{2} \\ a\nu & a & 0 \end{bmatrix} \begin{bmatrix} -a & 0 & a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & -a & 0 & a & a & 0 \end{bmatrix} =$$

$$= \frac{Eh}{2\cancel{a^2}(1-\nu^2)} \begin{bmatrix} \cancel{a^2}\left(1+\frac{1-\nu}{2}\right) & \cancel{a^2}\left(\nu+\frac{1-\nu}{2}\right) & -\cancel{a^2} & -\cancel{a^2}\frac{1-\nu}{2} & -\cancel{a^2}\frac{1-\nu}{2} & -\cancel{a^2}\nu \\ \cancel{a^2}\left(\nu+\frac{1-\nu}{2}\right) & \cancel{a^2}\left(1+\frac{1-\nu}{2}\right) & -\cancel{a^2}\nu & -\cancel{a^2}\frac{1-\nu}{2} & -\cancel{a^2}\frac{1-\nu}{2} & -\cancel{a^2} \\ -\cancel{a^2} & -\cancel{a^2}\nu & \cancel{a^2} & 0 & 0 & \cancel{a^2}\nu \\ -\cancel{a^2}\cdot\frac{1-\nu}{2} & -\cancel{a^2}\cdot\frac{1-\nu}{2} & 0 & \cancel{a^2}\frac{1-\nu}{2} & \cancel{a^2}\frac{1-\nu}{2} & 0 \\ -\cancel{a^2}\cdot\frac{1-\nu}{2} & -\cancel{a^2}\cdot\frac{1-\nu}{2} & 0 & \cancel{a^2}\frac{1-\nu}{2} & \cancel{a^2}\cdot\frac{1-\nu}{2} & 0 \\ -\cancel{a^2}\nu & -\cancel{a^2} & \cancel{a^2}\nu & 0 & 0 & \cancel{a^2} \end{bmatrix} =$$

$$= \frac{Eh}{(1-\nu^2)} \cdot \begin{bmatrix} \frac{3-\nu}{4} & \frac{1+\nu}{4} & -\frac{1}{2} & -\frac{1-\nu}{4} & -\frac{1-\nu}{4} & -\frac{\nu}{2} \\ \frac{1+\nu}{4} & \frac{3-\nu}{4} & -\frac{\nu}{2} & -\frac{1-\nu}{4} & -\frac{1-\nu}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\nu}{2} & \frac{1}{2} & 0 & 0 & \frac{\nu}{2} \\ -\frac{1-\nu}{4} & -\frac{1-\nu}{4} & 0 & \frac{1-\nu}{4} & \frac{1-\nu}{4} & 0 \\ -\frac{1-\nu}{4} & -\frac{1-\nu}{4} & 0 & \frac{1-\nu}{4} & \frac{1-\nu}{4} & 0 \\ -\frac{\nu}{2} & -\frac{1}{2} & \frac{\nu}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} =$$

$$\underline{K}^{e}_{tri_{\nu}} = \begin{bmatrix} \frac{(3-\nu)Eh}{4(1-\nu^2)} & \frac{Eh}{4(1-\nu)} & -\frac{Eh}{2(1-\nu^2)} & -\frac{Eh}{4(1+\nu)} & -\frac{Eh}{4(1+\nu)} & -\frac{Eh\nu}{2(1-\nu^2)} \\ \frac{Eh}{4(1-\nu)} & \frac{(3-\nu)Eh}{4(1-\nu^2)} & -\frac{Eh\nu}{2(1-\nu^2)} & -\frac{Eh}{4(1+\nu)} & -\frac{Eh}{4(1+\nu)} & -\frac{Eh}{2(1-\nu^2)} \\ -\frac{Eh}{2(1-\nu^2)} & -\frac{Eh\nu}{2(1-\nu^2)} & \frac{Eh}{2(1-\nu^2)} & 0 & 0 & \frac{Eh\nu}{2(1-\nu^2)} \\ -\frac{Eh}{4(1+\nu)} & -\frac{Eh}{4(1+\nu)} & 0 & \frac{Eh}{4(1+\nu)} & \frac{Eh}{4(1+\nu)} & 0 \\ -\frac{Eh}{4(1+\nu)} & -\frac{Eh}{4(1+\nu)} & 0 & \frac{Eh}{4(1+\nu)} & \frac{Eh}{4(1+\nu)} & 0 \\ -\frac{Eh\nu}{2(1-\nu^2)} & -\frac{Eh}{2(1-\nu^2)} & \frac{Eh\nu}{2(1-\nu^2)} & 0 & 0 & \frac{Eh}{2(1-\nu^2)} \end{bmatrix}$$

In this case the difference with the bar structure is more evident. There are more non-zero element than before, so that means that this model of structure is more precise, because there are more components that can give us a non-zero force. About the elements similar to the $\underline{K}^e_{bar}$ we can make the same consideration, but changing the relations obviously:

- $\boxed{\frac{A}{a} = \frac{(3-\nu)h}{4(1-\nu^2)}}$
- $\boxed{\frac{A_3\sqrt{2}}{4a} = \frac{h}{4(1-\nu^2)}}$

Also In this case we can see that the response of the structure due to "for" dof is completely different and more accurate than the bars structure: the "MIXED" elements are $\emptyset$ mostly in the $\underline{K}_{e_{bar}}$ while in the triangle are non-zero values.