Computational Structural Mechanics and Dynamics - Homework 3

Martin Vee Akselsen - Erasmus student

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1 Introduction

This report describes the solution of Assignement 3 in the subject Computational Structural Mechanics and Dynamics.

Contents

2 Exercise 1

All the calculations is done by hand and attached at the end of this exercise. In this report only the results and some important calculations will be shown.

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \mu = G = \frac{E}{2(1+\nu)} \tag{1}$$

2.1 1.1

Two equations, rewriting equation two as $E = 2\mu(1+\upsilon)$ and inserting it into the first equation for lambda we obtain the inverse relation for υ.

$$\nu = \frac{\lambda}{2(\mu+\lambda)} \tag{2}$$

We could now express E with μ and λ by inserting equation 2.

$$E = \mu(\frac{2\mu + 3\lambda}{\mu + \lambda}) \tag{3}$$

2.2 1.2

The elasticity matrix for plane stress expressed with E and ν is;

$$\text{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Inserting equation 2 and 3 into the matrix above we could express the matrix in terms of λ and μ as.

$$\text{E} = \begin{bmatrix} \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} & \frac{2\mu\lambda}{2\mu+\lambda} & 0 \\ \frac{2\mu\lambda}{2\mu+\lambda} & \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

The elasticity matrix for plane strain expressed with E and ν is;

$$\text{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{(1-\nu)} & 0 \\ \frac{\nu}{(1-\nu)} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

Inserting equation 2 and 3 into the matrix above we could express the matrix in terms of λ and μ as.

$$\text{E} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

2.3 1.3

The stress/strain matrix E for plain stress split in two matrixes, one for λ and one for μ

$$E = E_\lambda + E_\mu = \begin{bmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{4}$$

2.4 1.4

E_λ and E_μ expressed in terms of E and ν

$$E = E_\lambda + E_\mu = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{E}{2(1+\nu)}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

EXERCISE 4

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad (I) \qquad \mu = G = \frac{E}{2(1+\nu)} \quad (II)$$

(1)

$\Rightarrow$ (II) $\quad E = 2\mu(1+\nu)$

(II) in (I):

$$\lambda = \frac{2\mu\nu\cancel{(1+\nu)}}{\cancel{(1+\nu)}(1-2\nu)} \Rightarrow \nu = \frac{\lambda}{2\mu(1+\frac{\lambda}{\mu})} = \underline{\frac{\lambda}{2(\mu+\lambda)}}$$

(II)

$$E = 2\mu\left(1+\frac{\lambda}{2(\mu+\lambda)}\right) = \mu\left(\frac{2(\mu+\lambda)+\lambda}{(\mu+\lambda)}\right)$$

$$\Rightarrow \underline{E = \mu\left(\frac{2\mu+3\lambda}{\mu+\lambda}\right)}$$

(2) Stress: $\quad \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

$$\rightsquigarrow \frac{\mu\left(\frac{2\mu+3\lambda}{\mu+\lambda}\right)}{1-\left(\frac{\lambda}{2(\mu+\lambda)}\right)^2}\begin{bmatrix} 1 & \frac{\lambda}{2(\mu+\lambda)} & 0 \\ \frac{\lambda}{2(\mu+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{1-\frac{\lambda}{2(\mu+\lambda)}}{2} \end{bmatrix}$$

$$= \frac{4\mu(2\mu+3\lambda)(\mu+\lambda)}{4(\mu+\lambda)^2-\lambda^2} = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}$$

$$\Rightarrow E^{stress} = \begin{bmatrix} \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} & \frac{2\lambda\mu}{2\mu+\lambda} & 0 \\ \frac{2\lambda\mu}{2\mu+\lambda} & \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Strain:

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix}$$

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{\mu\left(\frac{2\mu+3\lambda}{\mu+\lambda}\right)\left(1-\frac{\lambda}{2(\mu+\lambda)}\right)}{\left(1+\frac{\lambda}{2(\mu+\lambda)}\right)\left(1-2\frac{\lambda}{2(\mu+\lambda)}\right)}$$

$$= \frac{\mu(2\mu+3\lambda)\left((\mu+\lambda)-\frac{\lambda}{2}\right)}{\left((\mu+\lambda)+\frac{\lambda}{2}\right)\left((\mu+\lambda)-\frac{\lambda}{2}\right)} = \frac{\mu(2\mu+3\lambda)\left(\mu+\frac{\lambda}{2}\right)}{\mu\left(\mu+\frac{3\lambda}{2}\right)}$$

$$* = 2\mu+\lambda$$

Multiplying with

$$\frac{\nu}{1-\nu} = \frac{\frac{\lambda}{2(\mu+\lambda)}}{1-\frac{\lambda}{2(\mu+\lambda)}}\cdot(2\mu+\lambda)* = \frac{1}{\frac{2(\mu+\lambda)}{\lambda}-1} = \frac{(2\mu+\lambda)\lambda}{(2(\mu+\lambda)-\lambda)}$$

$$= \underline{\underline{\lambda}}$$

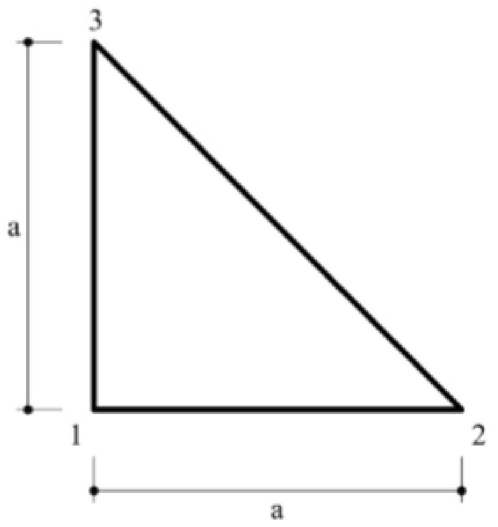
$$\frac{1-2\nu}{(2(1-\nu))} = \frac{1-2\frac{\lambda}{2(\mu+\lambda)}}{2\left(1-\frac{\lambda}{2(\mu+\lambda)}\right)} = \frac{1-\frac{\lambda}{\mu+\lambda}}{2-\frac{\lambda}{\mu+\lambda}}$$

Multiplied with *

$$= \frac{\frac{\mu+\lambda}{\mu+\lambda}-\frac{\lambda}{\mu+\lambda}}{\frac{2(\mu+\lambda)}{\mu+\lambda}-\frac{\lambda}{\mu+\lambda}}\cdot(2\mu+\lambda) = \underline{\underline{\mu}}$$

$$\Rightarrow E^{strain} = \begin{bmatrix}2\mu+\lambda & \lambda & 0\\ \lambda & 2\mu+\lambda & 0\\ 0 & 0 & \mu\end{bmatrix}$$

3 Exercise 2



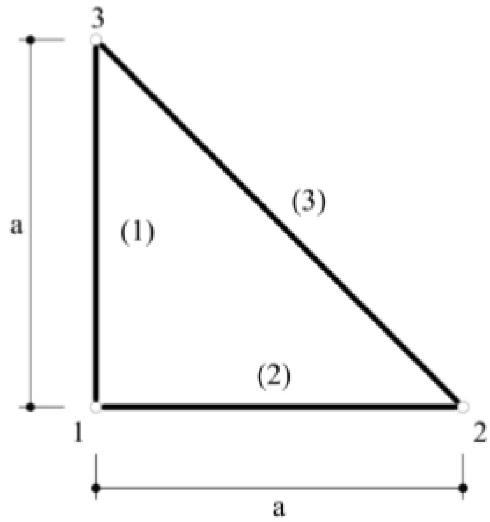


Figure 1: The two discrete structural models. To the left a plane linear Turner triangle and to the right a set of bars element.

The thickness h and length a is equal to 1 in this exercise. For the discrete structural model consisting of bars element one and two has the same cross section area.

3.1 2.1 Calculations of stiffness matrices

All calculations have been done with the use of Matlab, and the script will be attached after each subsection.

3.1.1 Turner triangle

To calculate the stiffness matrix for the Turner triangle I have used the relationship given in class using the kinematic equations and constitutive equations for plane stress giving us.

$$K^e = \int_{\Omega^e} hB^T DBd\Omega \tag{6}$$

Since the thickness throughout the whole area, D and B are constant we could write this as.

$$\mathbf{K}^e = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$

Calculating the matrix multiplication with the use of Matlab we obtain the stiffness matrix for the Turner triangle.

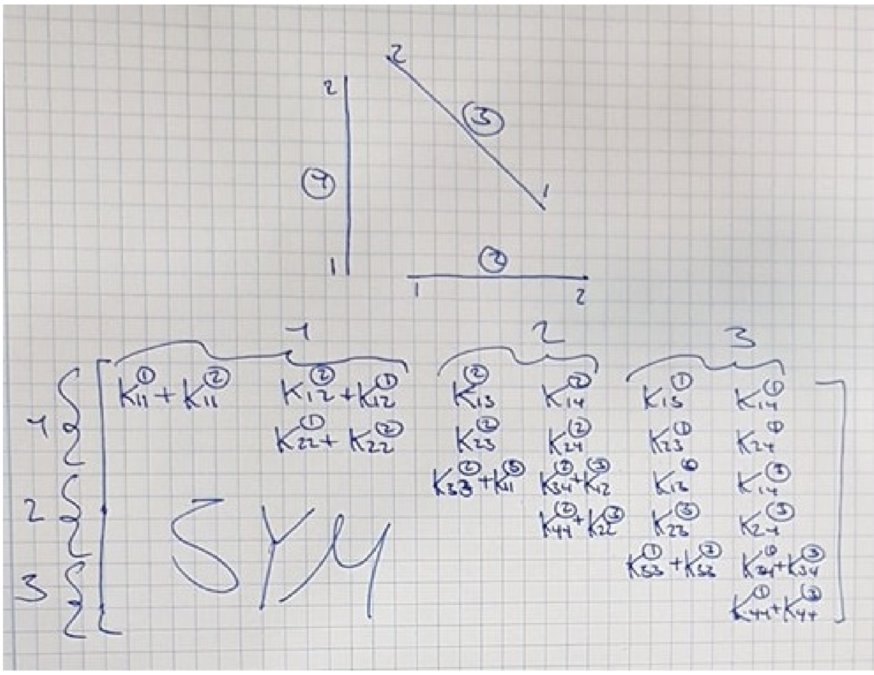
$$
\mathrm{K}_{tri} = \frac{E}{4} \begin{bmatrix} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}
$$

3.2 Three bars

To calculate the stiffness matrix I have found the local stiffness matrix for each bar, and then transformed them into global coordinates before assembled them into one global stiffness matrix for the whole structural model. Below the local stiffness matrix and the transformation matrix is shown.

$$\mathrm{K}_{local} = \frac{EA^e}{L^e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and T } = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

Below is the assembly of the local stiffness matrices.



Using this assembly we obtain from matlab that the stiffness matrix for this discrete structural model with the use of three bars is.

$$\mathrm{K}_{bars} = \frac{E}{4}\begin{bmatrix} 4A_2 & 0 & -4A_2 & 0 & 0 & 0 \\ 0 & 4A_1 & 0 & 0 & 0 & -4A_1 \\ -4A_2 & 0 & \sqrt{2}A_3+\mathrm{A}_2 & -\sqrt{2}A_3 & -\sqrt{2}A_3 & \sqrt{2}A_3 \\ 0 & 0 & -\sqrt{2}A_3 & \sqrt{2}A_3 & \sqrt{2}A_3 & -\sqrt{2}A_3 \\ 0 & 0 & -\sqrt{2}A_3 & \sqrt{2}A_3 & \sqrt{2}A_3 & -\sqrt{2}A_3 \\ 0 & -4A_1 & -\sqrt{2}A_3 & -\sqrt{2}A_3 & -\sqrt{2}A_3 & \sqrt{2}A_3+\mathrm{A}_1 \end{bmatrix}$$

```
clear;

%The local stiffness matrix that holds for the bar element 2.
Ke2 = [1 0 -1 0;0 0 0 0;-1 0 1 0;0 0 0 0];

%For bar 1 with a transformation angle of 90 degrees.
c1 = 0;
s1 = 1;
T1 = [c1 s1 0 0; -s1 c1 0 0; 0 0 c1 s1; 0 0 -s1 c1];
T1_trans = transpose(T1);
Ke1 = T1_trans*Ke2*T1;

%For bar 3 with a transformation angle of 135 degrees (or minus 45 degrees).
c3 = -1/sqrt(2);
s3 = 1/sqrt(2);
T3 = [c3 s3 0 0; -s3 c3 0 0; 0 0 c3 s3; 0 0 -s3 c3];
T3_trans = transpose(T3);
Ke3 = 1/(sqrt(2))*T3_trans*Ke2*T3; %With length equal sqrt(2)

%Assemblying the global stiffness matrix.
K_global = [Ke1(1,1)+Ke2(1,1) Ke1(1,2)+Ke2(1,2) Ke2(1,3) Ke2(1,4) Ke1(1,3) Ke1(1,4);...
    Ke1(2,1)+Ke2(2,1) Ke1(2,2)+Ke2(2,2) Ke2(2,3) Ke2(2,4) Ke1(2,3) Ke1(2,4);...
    Ke2(3,1) Ke2(3,2) Ke2(3,3)+Ke3(1,1) Ke2(3,4)+Ke3(1,2) Ke3(1,3) Ke3(1,4);...
    Ke2(4,1) Ke2(4,2) Ke2(4,3)+Ke3(2,1) Ke2(4,4)+Ke3(2,2) Ke3(2,3) Ke3(2,4);...
    Ke1(3,1) Ke1(3,2) Ke3(3,1) Ke3(3,2) Ke1(3,3)+Ke3(3,3) Ke1(3,4)+Ke3(3,4);...
    Ke1(4,1) Ke1(4,2) Ke3(4,1) Ke3(4,2) Ke1(4,3)+Ke3(4,3) Ke1(4,4)+Ke3(4,4)];
```

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3.3 2.2

It is not possible to make the two stiffness matrices equivalent by changing the values for the cross sections. But is possible to change the values in a matter that makes the two stiffness matrices more similar. By comparing values in the two different equations its possible to create equations that finds values for the cross sections making the matrices more similar.
For example comparing the three biggest values in the Turner stiffness matrices corresponding to the same coordinate in the stiffness matrix for the bars, and thereby calculate values for the cross section making the matrices more similar.

3.4 2.3

The two matrices are not similar because the two discrete structure models contributes to the stiffness in different ways. For the Turner triangle, the whole surface contributes to the stiffness in the directions, while for the bars it is only the bars that contributes to the stiffness. That means that a point on the surface for the second structural model would not contribute to any stiffness. Therefor, the two matrices cant have the same stiffness matrix.

3.5 2.4

If we dont consider ν as zero, the stiffness matrix for the Turner triangle will change, giving a lot of less coordinates with 0 in the matrix. This means that that more coordinates are multiplied to the displacement vector that impact the value of the force vector.