UNIVERSITAT POLITÈCNICA DE CATALUNYA

MASTER IN COMPUTATION MECHANICS AND NUMERICAL METHODS IN ENGINEERING

COMPUTATIONAL STRUCTURAL MECHANICS AND DYNAMICS

Assignment 4

by

Renan Alessio

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1- Introduction

The goal of the assignment is to analyze the isoparametric representation of 1D elements and structures of revolution and apply their formulations. A discussion on both subjects was also considered.

2 – Assignment 4.1

2.1 – Part 1

To obtain the Jacobian of the element presented in the assignment [1], a relationship between the cartesian coordinate x and the natural coordinate ζ needs to be found. For such task, it is possible to parametrize x as a function of ζ with a second-degree polynomial since the presented element has 3 three nodes [1]. The second-degree polynomial is the following:

$$x = \alpha_0 + \alpha_1\zeta + \alpha_2\zeta^2 \tag{1}$$

Considering that when x = 0 → ζ = -1, Equation 1 can be rewritten as:

$$0 = \alpha_0 + -\alpha_1 + \alpha_2 \tag{2}$$

Considering that when x = L → ζ = 1, Equation 1 can be rewritten a

$$L = \alpha_0 + \alpha_1 + \alpha_2 \tag{3}$$

Considering that when x = L/2 + αL → ζ = 0, Equation 1 can be rewritten as

$$\mathrm{L}/2 + \alpha\mathrm{L} = \alpha_0 \tag{4}$$

Equations 2-4 build a system which can be solved for the coefficients α_0, α_1 and α_2. The values of the coefficients are:

$$\alpha_0 = \mathrm{L}/2 + \alpha\mathrm{L}, \alpha_1 = L/2, \alpha_2 = -L\alpha$$

Replacing the values of the coefficients in Equation 1, the parametrization of x in terms of ζ becomes:

$$x = \frac{L}{2} + \alpha L + \frac{L}{2}\zeta - \alpha L\zeta^2 \tag{5}$$

With Equation (5), it is possible to calculate the Jacobian J:

$$J = \frac{dx}{d\zeta} = \frac{L}{2} - 2\alpha L\zeta \tag{6}$$

Considering the limits 1/4 < α < 1/4 with -1≤ ζ ≤ 1, Equation 6 will always be positive because the term 2αLζ, for any value of ζ in [-1,1], will not be greater than L/2. Hence, J will always be positive in this case. Considering α=0, Equation (6) takes the value of L/2 over the whole element.

2.1 – Part 2

To calculate the strain matrix **B**, the following equation is employed:

$$\boldsymbol{B} = J^{-1}\frac{d\boldsymbol{N}}{d\zeta}, where\ \boldsymbol{N} = [N_1 N_2 N_3] \tag{7}$$

N_i are the shape functions in natural coordinate ζ and are defined as [2]:

$$N_1 = \frac{\zeta^2}{2} - \frac{\zeta}{2}$$

$$N_2 = 1 + \zeta^2$$

$$N_3 = \frac{\zeta^2}{2} + \frac{\zeta}{2}$$

Calculating J^{-1} and the derivatives of the shape functions N_i w.r.t ζ, the **B** matrix is defined as:

$$\boldsymbol{B} = \left[\frac{2\zeta - 1}{L(1-4\alpha)}, \quad \frac{-4\zeta}{L(1-4\alpha)}, \quad \frac{1 + 2\zeta}{L(1-4\alpha)}\right]$$

3 – Discussion on Isoparametric Representation

The parametrization of geometric shape functions and displacement shape functions is very useful when higher order displacement shape functions are considered, or the discretized domain requires more accurate representation. Such advantage is due to the simplification of the integration limits, always from -1 to 1 in such case, and the facilitated employment of numerical integration, such as the Gauss Quadrature, during the computation of stiffness matrix and force vector. Moreover, if the same shape functions are applied for the displacement and the geometry, as known as isoparametric representation, only one set of shape functions is needed to interpolate both the geometry and the displacement field. Also, in 2D/3D analysis the parametrization of the geometric shape functions allows an evaluation of the element quality. Such study can be achieved through the Jacobian Matrix **J**, where the elements in the computational domain are “compared” with the reference element in the natural coordinate system. The coefficients in the Jacobian Matrix **J** can be used to measure the distortion of the elements in the computational domain.

4 – Assignment 4.2

4.1 – Part 1

The stiffness matrix for a 3-noded triangular element in axisymmetric case can be computed through the element stiffness submatrix according to the following equation [2]:

$$\boldsymbol{K}_{ij}^{(e)} = = \frac{\pi}{2A^{(e)2}} \iint_{A^{(e)}} \begin{bmatrix} \left(E_{11}b_ib_j + E_{44}c_ic_j\right)r + 2A^{(e)}\left(E_{13}b_iN_j + E_{31}b_jN_i\right) + 4A^{(e)2}E_{33}\frac{N_iN_j}{r} & \left(E_{12}b_ic_j + E_{44}c_ib_j\right)r + 2A^{(e)}E_{32}c_jN_i \\ \left(E_{21}c_ib_j + E_{44}b_ic_j\right)r + 2A^{(e)}E_{23}c_iN_j & \left(E_{22}c_ic_j + E_{44}b_ib_j\right)r \end{bmatrix} drdz \qquad (8)$$

where i,j and k = 1, 2 and 3.

The coefficients E_{ij} are the elements of the constitutive matrix **E** provided in the assignment [1]. The coefficients a_i, b_i and c_i are the coefficients of the shape function N_i and are defined below according to the node coordinates:

$$a_i = x_jy_k - x_ky_j, b_i = y_j - y_i, c_i = x_k - x_j; \qquad (9)$$

And the shape function is defined as follows:

$$N_i = \frac{1}{2A^{(e)}}(a_i + b_ix + c_iy), \qquad i = 1,2,3 \qquad (10)$$

Applying the data provided in the assignment [1] to Equation (8), the element stiffness $\mathbf{K}^{(e)}$ takes the form:

$$\boldsymbol{K}^{(e)} =$$

$$= \begin{bmatrix} \frac{4\pi Eb}{3} & 0 & -\frac{\pi Eb}{2} & 0 & \frac{\pi Eb}{6} & 0 \\ 0 & \frac{\pi Eb}{3} & \frac{\pi Ea}{3} & -\frac{\pi Eb}{3} & -\frac{\pi Ea}{3} & 0 \\ -\frac{\pi Eb}{2} & \frac{\pi Ea}{3} & \frac{2\pi}{a^2b^2}\left(\frac{11Ea^2b^3}{18} + \frac{Ea^4b}{6}\right) & -\frac{\pi Ea}{3} & \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{18} - \frac{Ea^4b}{6}\right) & 0 \\ 0 & -\frac{\pi Eb}{3} & -\frac{\pi Ea}{3} & \frac{\pi E}{3}\left(\frac{2a^2}{b} + b\right) & \frac{\pi Ea}{3} & -\frac{2\pi Ea^2}{3b} \\ \frac{\pi Eb}{6} & -\frac{\pi Ea}{3} & \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{18} - \frac{Ea^4b}{6}\right) & \frac{\pi Ea}{3} & \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{9} + \frac{Ea^4b}{6}\right) & 0 \\ 0 & 0 & 0 & -\frac{2\pi Ea^2}{3b} & 0 & \frac{2\pi Ea^2}{3b} \end{bmatrix}$$

4.2 – Part 2

From the stiffness matrix $\mathbf{K}^{(e)}$ computed in the section above, it is noticeable that the even columns and rows produce zero column and row vector. This is due to the equilibrium imposed in the z-direction among the nodes in the element and to the fact that the circumferential strain and stress do not depend on that coordinate. Below, the sum of rows 2, 4 and 6 is presented and well as the sum of the columns 2, 4 and 6:

<u>Sum of rows 2, 4 and 6:</u>

$$\begin{bmatrix} 0 & \frac{\pi Eb}{3} & \frac{\pi Ea}{3} & -\frac{\pi Eb}{3} & -\frac{\pi Ea}{3} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\frac{\pi Eb}{3} & -\frac{\pi Ea}{3} & \frac{\pi E}{3}\left(\frac{2a^2}{b} + b\right) & \frac{\pi Ea}{3} & -\frac{2\pi Ea^2}{3b} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & -\frac{2\pi Ea^2}{3b} & 0 & \frac{2\pi Ea^2}{3b} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Sum of columns 2, 4 and 6:</u>

$$\begin{bmatrix} 0 & \frac{\pi Eb}{3} & \frac{\pi Ea}{3} & -\frac{\pi Eb}{3} & -\frac{\pi Ea}{3} & 0 \end{bmatrix}^T$$

$$+\left[0 \quad -\frac{\pi E b}{3} \quad -\frac{\pi E a}{3} \quad \frac{\pi E}{3}\left(\frac{2a^2}{b}+b\right) \quad \frac{\pi E a}{3} \quad -\frac{2\pi E a^2}{3b}\right]^T$$

$$+\left[0 \quad 0 \quad 0 \quad -\frac{2\pi E a^2}{3b} \quad 0 \quad \frac{2\pi E a^2}{3b}\right]^T$$

$$= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

Nevertheless, the sum of columns and rows 1, 3 and 5 do not add up to zero. This is due to the fact that the radial displacement is responsible for the strain and stress in the circumferential direction. Therefore, the non-zero values are related to the internal energy term in the circumferential coordinate which does not have a conjugate (external work done in that direction). The sum of rows and columns 1, 3 and 5 are presented below as to show such results:

<u>Sum of rows 1, 3 and 5:</u>

$$\left[\frac{4\pi E b}{3} \quad 0 \quad -\frac{\pi E b}{2} \quad 0 \quad \frac{\pi E b}{6} \quad 0\right]$$

$$+\left[-\frac{\pi E b}{2} \quad \frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{11Ea^2b^3}{18}+\frac{Ea^4b}{6}\right) \quad -\frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{18}-\frac{Ea^4b}{6}\right) \quad 0\right]$$

$$+\left[\frac{\pi E b}{6} \quad -\frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{18}-\frac{Ea^4b}{6}\right) \quad \frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{9}+\frac{Ea^4b}{6}\right) \quad 0\right]$$

$$=\left[\pi E b \quad 0 \quad \frac{4\pi}{a^2b^2}\left(\frac{Ea^2b^3}{3}\right)-\frac{\pi E b}{2} \quad 0 \quad \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{6}\right)+\frac{\pi E b}{6} \quad 0\right]$$

<u>Sum of columns 1, 3 and 5:</u>

$$\left[\frac{4\pi E b}{3} \quad 0 \quad -\frac{\pi E b}{2} \quad 0 \quad \frac{\pi E b}{6} \quad 0\right]^T$$

$$+\left[-\frac{\pi E b}{2} \quad \frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{11Ea^2b^3}{18}+\frac{Ea^4b}{6}\right) \quad -\frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{18}-\frac{Ea^4b}{6}\right) \quad 0\right]^T$$

$$+\left[\frac{\pi E b}{6} \quad -\frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{18}-\frac{Ea^4b}{6}\right) \quad \frac{\pi E a}{3} \quad \frac{2\pi}{a^2b^2}\left(\frac{Ea^2b^3}{9}+\frac{Ea^4b}{6}\right) \quad 0\right]^T$$

$$= \left[\pi E b \quad 0 \quad \frac{4\pi}{a^2 b^2}\left(\frac{E a^2 b^3}{3}\right) - \frac{\pi E b}{2} \quad 0 \quad \frac{2\pi}{a^2 b^2}\left(\frac{E a^2 b^3}{6}\right) + \frac{\pi E b}{6} \quad 0\right]^T$$

4.3 – Part 3

To compute the consistent force vector according to the data provided in the assignment [1], the following equation can be employed [2]:

$$\boldsymbol{f}^{(e)} = \frac{\pi A^{(e)}}{6}\begin{bmatrix}(2r_i + r_j + r_k)b_r \\ (2r_i + r_j + r_k)b_z \\ (r_i + 2r_j + r_k)b_r \\ (r_i + 2r_j + r_k)b_z \\ (r_i + r_j + 2r_k)b_r \\ (r_i + r_j + 2r_k)b_z\end{bmatrix} \tag{11}$$

Applying the data from the assignment [1] to Equation 11, the consistent force vector takes the following form:

$$\boldsymbol{f}^{(e)} = \frac{\pi a b}{12}\begin{bmatrix}0 \\ -2ag \\ 0 \\ -3ag \\ 0 \\ -3ag\end{bmatrix}$$

5 – Discussion on Structures of Revolution

Structures of revolution are originally 3D structures, but since they have a revolution axis, they can be reduced to a 2D framework under certain conditions. Such conditions are the boundary conditions which must be independent of the circumferential coordinate and symmetric according to the revolution axis. If these conditions are met, the problem is greatly reduced and a 2D analysis can be performed. The simplification brings reduction to computational time and enables a more detailed study on mesh convergence under the same computational limitations. Nevertheless, it is worth mentioning that the formulation of stiffness matrix and force vector for elements in axisymmetric conditions are different from the 2D elasticity case studied in the previous assignment. This is due to the non-zero values of ε_θ and σ_θ, which arise from the displacement in the radial direction. Therefore, differently from the 2D elasticity case, the internal energy related to the suppressed coordinate is different than zero.

6 – References

[1] – Assignment-4, Computational Structural Mechanics and Dynamics, Master of Science in Computational Mechanics, 2020.

[2] – Oñate, E., "Structural Analysis with the Finite Element Method – Linear Statics", vol. 1, 2008.