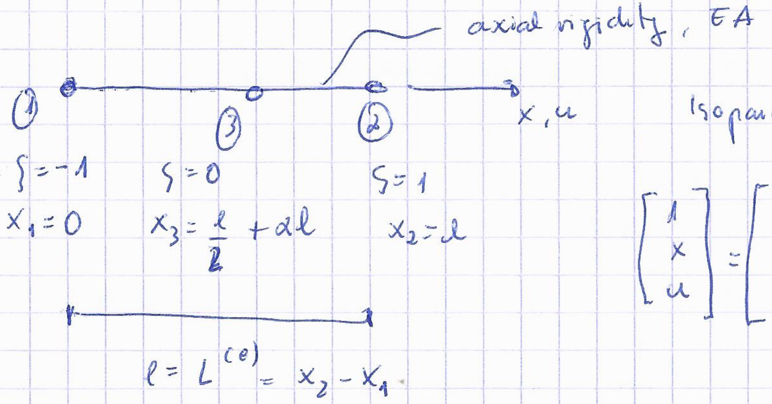
Assignment 4.1 — 09/03/2020

On "Isoparametric representation"



Isoparametric definition

$$\begin{bmatrix}1\\ x\\ u\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ u_1 & u_2 & u_3\end{bmatrix}\begin{bmatrix}N_1^e\\ N_2^e\\ N_3^e\end{bmatrix}$$

1) Get the Jacobian $\Rightarrow J = \dfrac{dx}{d\xi} = f(l, \alpha, \xi)$

1.1) if $-\dfrac{1}{4} < \alpha < \dfrac{1}{4} \Rightarrow J > 0$ over $-1 \leq \xi \leq 1$

1.2) $\alpha = 0 \Rightarrow J = \dfrac{l}{2} \Rightarrow$ cte over element.

2.) $\underline{B}$

1) First, we get the quadratic shape functions defined on Lagrange interpolation functions of 2nd order.

$$N_1^e(\xi) = \frac{\xi(\xi-1)}{2} \qquad N_2^e(\xi) = \frac{\xi(\xi+1)}{2} \qquad N_3(\xi) = -(\xi+1)(\xi-1)$$

Using the above trial functions into the isoparametric definition, we find $\bar{x}$ in function of ξ

$$\bar{x} = \bar{x}_1 N_1^e(\xi) + \bar{x}_2 N_2^e(\xi) + \bar{x}_3 N_3^e(\xi)$$, substituting $N_1^e(\xi), N_2^e(\xi), N_3^e(\xi)$ and x_1, x_2, x_3 given before,

$$\bar{x} = \frac{l}{2}\xi(\xi+1) + \left(\frac{l}{2} + \alpha l\right)\left(-(\xi+1)(\xi-1)\right) = \frac{l}{2}\xi(\xi+1) + \left(\frac{l}{2} + \alpha l\right)(1-\xi^2)$$

Since the Jacobian is defined on:

$$J = \frac{d\bar{x}}{d\xi} = \bar{x}_1 \frac{dN_1^e(\xi)}{d\xi} + \bar{x}_2 \frac{dN_2^e(\xi)}{d\xi} + \bar{x}_3 \frac{dN_3^e(\xi)}{d\xi}$$

$$= \bar{x}_1\left(\xi - \frac{1}{2}\right) + \bar{x}_2\left(\xi + \frac{1}{2}\right) - 2\xi\bar{x}_3$$

thus, the mapping between $\bar{x}$ and ξ leads to

$$J = \ell\left(\frac{1}{2} - 2\alpha\xi\right)$$

From here, we know that the Jacobian vanishes when $J=0$

$$0 = \ell\left(\frac{1}{2} - 2\alpha\xi\right) \Rightarrow \xi = \frac{1}{4\alpha}$$

thus, ~~in~~ the case in which $|\alpha| = \frac{1}{4}$, $\xi = \pm 1$, the Jacobian will vanish at one of the end points of the element.

1.1) if $-1 \leq \xi \leq 1$ when $-\frac{1}{4} < \alpha < \frac{1}{4} \Rightarrow J > 0$

1.2) if $\alpha = 0 \overset{\text{since}}{\Rightarrow} J = \ell\left(\frac{1}{2} - 2\alpha\xi\right) \Rightarrow J(\alpha=0) = \frac{\ell}{2}$

2)

we know that, $u = u_1 N_1 + u_2 N_2 + u_3 N_3$

From the given definition, $\varepsilon = \frac{du}{dx} \Rightarrow \varepsilon = u_1 \frac{dN_1}{d\bar{x}} + u_2 \frac{dN_2}{d\bar{x}} + u_3 \frac{dN_3}{d\bar{x}}$

By using the chain rule,

$$\frac{dN_i}{d\bar{x}} = \frac{dN_i}{d\xi}\frac{d\xi}{d\bar{x}} = \frac{dN_i}{d\xi} J^{-1}$$

and; therefore, calculating the strain as,

$$\varepsilon = \frac{1}{J}\left(u_1 \frac{dN_1}{d\xi} + u_2 \frac{dN_2}{d\xi} + u_3 \frac{dN_3}{d\xi}\right) \rightarrow$$ ~~in~~ the case in which $J=0$ would generate (axial) strain to reach infinity.

$\underline{\varepsilon} = \underline{\underline{B}}\,\underline{u}^e$ where $B = \frac{dN}{dx} = \frac{1}{J}\frac{dN}{d\xi}$

as stated before. $\underline{N} = [N_1 \; N_2 \; N_3] = \left[\frac{\xi(\xi-1)}{2} \quad \frac{\xi(\xi+1)}{2} \quad (1-\xi^2)\right]$

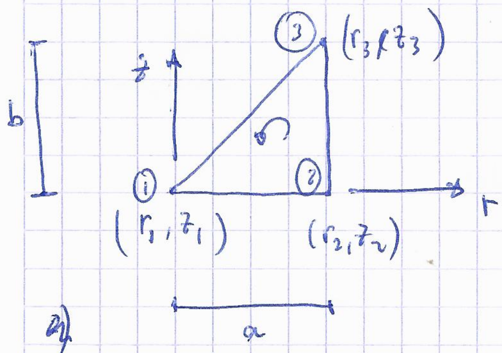
$$B = \frac{1}{\ell\left(\frac{1}{2} - 2\alpha\xi\right)}\left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right] =$$

$$= \left[-\frac{(2\xi-1)}{L(4\alpha\xi-1)} \quad -\frac{(2\xi+1)}{L(4\alpha\xi-1)} \quad \frac{4\xi}{L(4\alpha\xi-1)}\right]$$

"On structures of revolution"

Assignment 4.2

1) $\underline{K}^e$ for the following axisymmetric triangle



$(r_1, z_1) = (0,0)$

$(r_2, z_2) = (a, 0)$

$(r_3, z_3) = (a, b)$

Isotropic material:

$\nu = 0$

$$\underline{\underline{E}} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

2) $K^e(2,4,6) \to 0$

$K^e(1,3,5) \not\to 0$

3) $f^e \xrightarrow{\text{when}} \underline{b} = [0, -g]^T$

1) First we get the shape functions on every node

we know, from definition that,

$$N_i^{(e)} = \frac{1}{2A^{(e)}}(a_{1i} + a_{2i} r + a_{3i} z) \quad \text{for } i = 1,2,3$$

$$\to 2A^{(e)} = 2 \cdot \frac{ab}{2} = ab$$

where
$$\begin{cases} a_{1i} = r_j z_k - z_j r_k \\ a_{2i} = z_j - z_k \\ a_{3i} = r_k - r_j \end{cases}$$

1-2 $\Rightarrow$
$$\left.\begin{aligned} a_{13} &= r_1 z_2 - z_1 r_2 = 0 \\ a_{23} &= z_1 - z_2 = 0 \\ a_{33} &= r_2 - r_1 = a \end{aligned}\right\} \Rightarrow N_3^{(e)} = \frac{1}{ab} az = \frac{z}{b}$$

2-3 $\Rightarrow$
$$\left.\begin{aligned} a_{11} &= r_2 z_3 - z_2 r_3 = ab \\ a_{21} &= z_2 - z_3 = -b \\ a_{31} &= r_3 - r_2 = 0 \end{aligned}\right\} N_1^{(e)} = \frac{1}{ab}(ab - br) = \frac{a-r}{a}$$

3-1 $\Rightarrow$
$$\left.\begin{aligned} a_{12} &= r_3 z_1 - z_3 r_1 = 0 \\ a_{22} &= z_3 - z_1 = b \\ a_{32} &= r_1 - r_3 = -a \end{aligned}\right\} N_2^{(e)} = \frac{1}{ab}(br - az)$$

we know that

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{rz} & 0 \\ \varepsilon_{rz} & \varepsilon_{zz} & 0 \\ 0 & 0 & \varepsilon_{\theta\theta} \end{bmatrix} \Rightarrow \underline{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ 2\varepsilon_{rz} \end{bmatrix} = \underline{D}\,\underline{u}^{e} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ 1/r & 0 \\ \partial/\partial z & \partial/\partial r \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix}$$

No variation on θ direction since model is axisymmetric

and $\underline{u} = \underline{\underline{N}}\, u^{(e)}$, therefore, $\underline{\varepsilon} = \underline{\underline{D}}\,\underline{\underline{N}}\, u^{(e)} = \underline{\underline{B}}\,\underline{u}^{(e)}$

thus, $\underline{\underline{B}} = \underline{\underline{D}}\,\underline{\underline{N}}$, this implies calculating:

$$\underline{\underline{B}} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix}$$

and
$$\underline{\underline{N}} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

to get
$$\underline{\underline{K}}^{(e)} = \int_{\Omega^{(e)}} r\, \underline{\underline{B}}^{T}\, \underline{\underline{E}}\, \underline{\underline{B}}\; \partial\Omega^{(e)}$$

$$\underline{\underline{B}} = \begin{bmatrix} -\frac{1}{a} & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} & 0 & \frac{1}{b} \\ \frac{ab-br}{abr} & 0 & \frac{br-az}{abr} & 0 & \frac{z}{br} & 0 \\ 0 & -\frac{1}{a} & -\frac{1}{b} & \frac{1}{a} & \frac{1}{b} & 0 \end{bmatrix}$$

$$\underline{\underline{B}}^{T}\underline{\underline{E}} = E \begin{bmatrix} -\frac{1}{a} & 0 & \frac{ab-br}{abr} & 0 \\ 0 & 0 & 0 & -\frac{1}{2a} \\ \frac{1}{a} & 0 & \frac{br-az}{abr} & -\frac{1}{2b} \\ 0 & -\frac{1}{b} & 0 & \frac{1}{2a} \\ 0 & 0 & \frac{z}{br} & \frac{1}{2b} \\ 0 & \frac{1}{b} & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{B}}^T \underline{\underline{E}}\, \underline{\underline{B}} = E \begin{bmatrix} \frac{a^2-2ar+2r^2}{a^2r^2} & 0 & \frac{(ab-br)(br-az)}{a^2b^2r^2}-\frac{1}{a^2} & 0 & \frac{z(a-r)}{abr^2} & 0 \\ 0 & \frac{1}{2a^2} & \frac{1}{2ab} & -\frac{1}{2a^2} & -\frac{1}{2ab} & 0 \\ \frac{(ab-br)(br-az)}{a^2b^2r^2}-\frac{1}{a^2} & \frac{1}{2ab} & \frac{1}{2a^2}+\frac{1}{2b^2}+\frac{(br-az)^2}{a^2b^2r^2} & -\frac{1}{2ab} & \frac{z(br-az)}{ab^2r^2}-\frac{1}{2b^2} & 0 \\ 0 & -\frac{1}{2a^2} & -\frac{1}{2ab} & \frac{1}{2a^2}+\frac{1}{b^2} & \frac{1}{2ab} & -\frac{1}{b^2} \\ \frac{z(a-r)}{abr^2} & -\frac{1}{2ab} & \frac{z(br-az)}{ab^2r^2}-\frac{1}{2b^2} & \frac{1}{2ab} & \frac{r^2+2z^2}{2b^2r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{b^2} & 0 & \frac{1}{b^2} \end{bmatrix}$$

we know need to integrate too on to get

$$\underline{\underline{K}}^{(e)} = \int_{\Omega^{(e)}} r\, \underline{\underline{B}}^T \underline{\underline{E}}\, \underline{\underline{B}}\ \partial\Omega^{(e)}$$

where, the surface integrations domains are $z - z_1 = m(r - r_1)$

$$m = \frac{z_2 - z_1}{r_2 - r_1} = \frac{b-0}{a-0} \Rightarrow z = \frac{br}{a} \quad \text{in } dz$$

$$r = a \quad \text{in } \partial r$$

$$\underline{\underline{K}}^{(e)} = \int_0^a \int_0^{\frac{br}{a}} r\, \underline{\underline{B}}^T \underline{\underline{E}}\, \underline{\underline{B}}\ dz\, dr =$$

$$\underline{\underline{K}}^{(e)} = E \begin{bmatrix} \frac{2b}{3} & 0 & -\frac{b}{4} & 0 & \frac{b}{12} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \\ -\frac{b}{4} & \frac{a}{6} & \frac{a^2}{6b}+\frac{4b}{9} & -\frac{a}{6} & -\frac{a^2}{6b}+\frac{b}{18} & 0 \\ 0 & -\frac{b}{6} & -\frac{a}{6} & \frac{a^2}{3b}+\frac{b}{6} & \frac{a}{6} & -\frac{a^2}{3b} \\ \frac{b}{12} & -\frac{a}{6} & -\frac{a^2}{6b}+\frac{b}{18} & \frac{a}{6} & \frac{a^2}{6b}+\frac{b}{9} & 0 \\ 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

2) Sum of rows (and columns) 2, 4 and 6 of $\underline{\underline{K}}^e$ vanish, while sum of rows (and colums) 1, 3 and 5 of $\underline{\underline{K}}^e$ sum $\boxed{b}$

This is due to have, the element, a rigid body motion in the z-direction. The set of equations of $\underline{\underline{K}}^{(e)}$ belonging to rows and columns 2, 4, 6 describe the displacements in the z-direction of every node. Since the element is under rigid body motion conditions without any imposed external forces, the values in these rows/columns will have same values and the ~~matri~~ K^e matrix coefficients cancel each other out.

Oppositely, due to the imposition of axis-symmetry, there is no possibility fo rigid body motion in the r-direction and consequently, the coefficients do not cancel out. There exist non-zero internal forces due to radial deformation.

3) constant force vector f^e for $\underline{b} = [0, -g]^T$

From the definition, we know

$$f^{(e)} = \int_{\Omega^{(e)}} r \underline{\underline{N}}^T \underline{b} \, d\Omega^{(e)} \Rightarrow \oint f^{(e)} = \int_0^a \int_0^{\frac{br}{a}} r \underline{\underline{N}}^T \underline{b} \, dz \, dr$$

where

$$\underline{\underline{N}}^T := \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} = \begin{bmatrix} \frac{a-r}{a} & 0 \\ 0 & \frac{a-r}{a} \\ \frac{1}{ab}(br-az) & 0 \\ 0 & \frac{1}{ab}(br-az) \\ \frac{z}{b} & 0 \\ 0 & \frac{z}{b} \end{bmatrix}$$

$$f^{(e)} = \begin{bmatrix} 0 \\ -\frac{a^2bg}{12} \\ 0 \\ -\frac{a^2bg}{8} \\ 0 \\ -\frac{a^2bg}{8} \end{bmatrix}$$