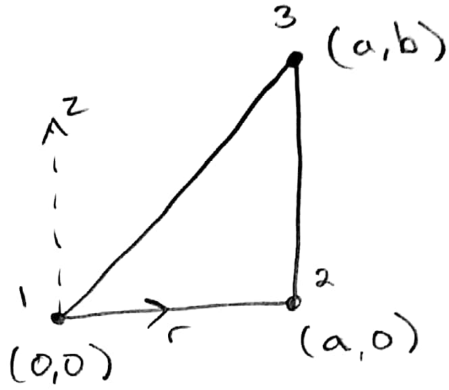
Alexander Keiser

CSMD Assignment # 4



4.1.1

$$E = C = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$A = \frac{1}{2}ab$

$f_1 = r_2 z_3 - r_3 z_2 = ab - 0$

$r_1 = 0 \qquad r_2 = a \qquad r_3 = a$

$f_2 = r_3 z_1 - r_1 z_3 = 0 - 0$

$z_1 = 0 \qquad z_2 = 0 \qquad z_3 = b$

$f_3 = r_1 z_2 - r_2 z_1 = 0 - 0$

$N_1 = 1 - \frac{r}{a}$

$N_2 = \frac{r}{a} - \frac{z}{b}$

$N_3 = \frac{z}{b}$

$$B^T = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ \frac{2AN_1}{r} & 0 & \frac{2AN_2}{r} & 0 & \frac{2AN_3}{r} & 0 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$b_1 = -b \qquad c_1 = 0$

$b_2 = b \qquad c_2 = -a$

$b_3 = 0 \qquad c_3 = a$

We will now evaluate K using matrix B at the centroid of the element

$$\boxed{K = 2\pi \bar{r} A \bar{B} C \bar{B}^T}$$

where $\bar{B} = B(\bar{r}, \bar{z})$ $\quad \bar{r} = \frac{r_1 + r_2 + r_3}{3} = \frac{2}{3}a$

$$\bar{z} = \frac{z_1 + z_2 + z_3}{3} = \frac{1}{3}b$$

now evaluating the shape functions at $\bar{r}, \bar{z}$ gives us the following result

$$N_1 = 1 - \frac{\bar{r}}{a} = 1 - \frac{(2/3 a)}{a} = \frac{1}{3}$$

$$N_2 = \frac{\bar{r}}{a} - \frac{\bar{z}}{b} = \frac{2}{3} - \frac{1}{3} = 1/3$$

$$N_3 = \frac{\bar{z}}{b} = \frac{1}{3}$$

now we can evaluate $\frac{2AN_1}{\bar{r}}, \frac{2AN_2}{\bar{r}}, \frac{2AN_3}{\bar{r}}$

$$\frac{2AN_{1,2,3}}{\bar{r}} = \frac{2(\frac{1}{2}ab)(1/3)}{(2/3 a)} = \frac{\frac{1}{3}ab}{\frac{2}{3}a} = \frac{1}{2}b$$

now we can evaluate $\bar{B}^T$

$$\bar{B}^T = \frac{1}{2(\frac{1}{2}ab)} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{1}{2}b & 0 & \frac{1}{2}b & 0 & \frac{1}{2}b & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

we will now form our K matrix on the next page using the formula

$$K = 2\pi \bar{r} A \bar{B} \bar{C} \bar{B}^T$$

first we will evaluate $\bar{B}C\bar{B}^T =$

$$\frac{E}{4A^2}\begin{bmatrix} -b & 0 & \frac{1}{2}b & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & \frac{1}{2}b & -a \\ 0 & -a & 0 & b \\ 0 & 0 & \frac{1}{2}b & a \\ 0 & a & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{1}{2}b & 0 & \frac{1}{2}b & 0 & \frac{1}{2}b & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix} = \bar{B}C\bar{B}^T$$

and reducing gives us

$$\begin{bmatrix} -b & 0 & \frac{1}{2}b & 0 \\ 0 & 0 & 0 & -b \\ b & 0 & \frac{1}{2}b & -a \\ 0 & -a & 0 & b \\ 0 & 0 & \frac{1}{2}b & a \\ 0 & a & 0 & 0 \end{bmatrix}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{1}{2}b & 0 & \frac{1}{2}b & 0 & \frac{1}{2}b & 0 \\ 0 & -\frac{b}{2} & -\frac{a}{2} & \frac{b}{2} & \frac{a}{2} & 0 \end{bmatrix} = \bar{B}C\bar{B}^T$$

reducing further gives us

$$\bar{B}C\bar{B}^T = \frac{E}{4A^2}\begin{bmatrix} \frac{5}{4}b^2 & 0 & -\frac{3}{4}b^2 & 0 & \frac{1}{4}b^2 & 0 \\ 0 & \frac{1}{2}b^2 & \frac{1}{2}ab & -\frac{1}{2}b^2 & -\frac{1}{2}ab & 0 \\ -\frac{3}{4}b^2 & \frac{1}{2}ab & \left(\frac{5}{4}b^2+\frac{1}{2}a^2\right) & -\frac{1}{2}ab & \left(\frac{1}{4}b^2-\frac{1}{2}a^2\right) & 0 \\ 0 & -\frac{1}{2}b^2 & -\frac{1}{2}ab & \frac{1}{2}b^2+a^2 & \frac{1}{2}ab & -a^2 \\ \frac{1}{4}b^2 & \frac{1}{2}ab & \left(\frac{1}{4}b^2-\frac{1}{2}a^2\right) & \frac{1}{2}ab & \frac{1}{4}b^2+\frac{1}{2}a^2 & 0 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \end{bmatrix}$$

And we can now form K matrix

$$K = 2\pi\left(\frac{2}{3}a\right) A \bar{B} C \bar{B}^T$$

$$K = \frac{2\pi E}{3b}\begin{bmatrix} \frac{5}{4}b^2 & 0 & -\frac{3}{4}b^2 & 0 & \frac{1}{4}b^2 & 0 \\ & \frac{1}{2}b^2 & \frac{1}{2}ab & -\frac{1}{2}b^2 & -\frac{1}{2}ab & 0 \\ & & \left(\frac{5}{4}b^2+\frac{1}{2}a^2\right) & -\frac{1}{2}ab & \left(\frac{1}{4}b^2-\frac{1}{2}a^2\right) & 0 \\ & & & \left(\frac{1}{2}b^2+a^2\right) & \frac{1}{2}ab & -a^2 \\ & \text{symmetric} & & & \left(\frac{1}{4}b^2+\frac{1}{2}a^2\right) & 0 \\ & & & & & a^2 \end{bmatrix}$$

4.1.2

We will now show that rows 2, 4, 6 vanish and explain why

Row 2: $0 + \frac{1}{2}b^2 + \frac{1}{2}ab - \frac{1}{2}b^2 - \frac{1}{2}ab + 0 = 0$

Terms cancel

Row 4: $0 - \frac{1}{2}b^2 - \frac{1}{2}ab + \left(\frac{1}{2}b^2 + a^2\right) + \frac{1}{2}ab - a^2 = 0$

Terms cancel

Row 6: $0 + 0 + 0 - a^2 + 0 + a^2 = 0$

Terms cancel

The physical meaning behind these rows summing to zero is because there is only allowed deformation in the radial (r) direction. There is no deformation in the z direction possible.

We will now show that rows 1, 3, 5 do not vanish and explain why

Row 1: $\frac{5}{4}b^2 + 0 - \frac{3}{4}b^2 + 0 + \frac{1}{4}b^2 + 0 = \frac{3}{4}b^2$

★ No terms cancel

Row 3: $-\frac{3}{4}b^2 + \frac{1}{2}ab + \left(\frac{5}{4}b^2 + \frac{1}{2}a^2\right) - \frac{1}{2}ab + \left(\frac{1}{4}b^2 - \frac{1}{2}a^2\right) + 0 = \frac{3}{4}b^2$

Terms Cancel   Terms cancel

Row 5: $\frac{1}{4}b^2 - \frac{1}{2}ab + \left(\frac{1}{4}b^2 - \frac{1}{2}a^2\right) + \frac{1}{2}ab + \left(\frac{1}{4}b^2 + \frac{1}{2}a^2\right) + 0 = \frac{3}{4}b^2$

Terms Cancel

The physical meaning behind this is that there will be allowed deformation in the r direction so it would not physical sense if these rows were zero

4.1.3

we will now compute the consistant nodal force vector with our shape functions from before

$$N_1 = 1 - \frac{r}{a}$$

$$N_2 = \frac{r}{a} - \frac{z}{b} \qquad N = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix}$$

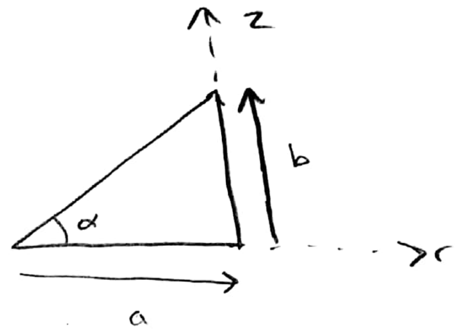
$$N_3 = \frac{z}{b} \qquad b = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

$$N^T b r = \begin{bmatrix} N_1 & 0 \\ N_2 & 0 \\ N_3 & 0 \\ 0 & N_1 \\ 0 & N_2 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -N_1 g r \\ -N_2 g r \\ -N_3 g r \end{bmatrix}$$

$$f_{ext} = \iint_{r\,z} N^T b r \, dz dr = \iint_{r\,z} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left(-gr + \frac{gr^2}{a}\right) \\ \left(-\frac{gr^2}{a} + \frac{grz}{b}\right) \\ \left(\frac{-grz}{b}\right) \end{bmatrix} dz dr$$

we will perform the integrals on the following page

Term 4



$$\tan\alpha = \frac{b}{a} = \frac{z}{r}$$

$$z = \frac{br}{a}$$

$$f^4_{ext} = \int_0^a \int_0^{br/a} \left(-gr + \frac{gr^2}{a}\right) dz\,dr$$

$$f^4_{ext} = \int_0^r \left(-grz + \frac{gr^2 z}{a}\right) dr$$

$$f^4_{ext} = \int_0^r \left(\frac{-gbr^2}{a} + \frac{gbr^3}{a^2}\right) dr$$

$$f^4_{ext} = \frac{-gba^3}{3a} + \frac{gba^4}{4a^2} = \frac{-gba^2}{3} + \frac{gba^2}{4}$$

$$f^4_{ext} = \frac{-4gba^2 + 3gba^2}{12}$$

$$f^4_{ext} = \frac{-gba^2}{12}$$

Term 5

$$f^5_{ext} = \int_0^a \int_0^{br/a} \left(\frac{-gr^2}{a} + \frac{grz}{b}\right) dz\,dr$$

$$f^5_{ext} = \int_0^a \left(\frac{-gr^2}{a}\left(\frac{br}{a}\right) + \frac{grb^2r^2}{2ba^2}\right) dr$$

$$f^5_{ext} = \frac{-ga^4b}{4a^2} + \frac{gr^4b^2}{8ba^2} = \frac{-ga^2b}{4} + \frac{ga^2b}{8}$$

$$f^5_{ext} = \frac{-ga^2b}{8}$$

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Term 6

$$f_{ext}^{6} = \int_0^a \int_0^{br/a} \left(-\frac{grz}{b}\right) dz\, dr$$

$$f_{ext}^{6} = \int_0^a \left(\frac{-grb^2 r^2}{2ba^2}\right) dr$$

$$f_{ext}^{6} = \frac{-gr^4 b^2}{4 \cdot 2ba^2} \Big|_0^a$$

$$f_{ext}^{6} = \frac{-ga^2 b}{8}$$

returning to the vector and plugging our values gives us...

$$f_{ext} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-gba^2}{12} \\ \frac{-gba^2}{8} \\ \frac{-gba^2}{8} \end{bmatrix}$$