

Assignment 4 - Structures of Revolution

Luan Malikoski Vieira

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4.1

1 - Computation of elemental Stiffness Matrix K^e

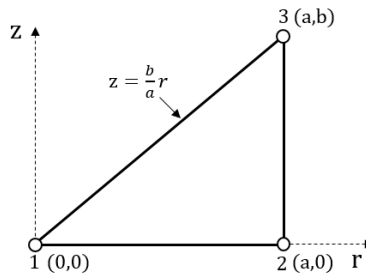


Figure 1: Axisymmetric triangle

For a linear triangle element (2D), the shape functions in terms of global coordinates are given by equation (1). Where r a z takes the place of x and y respectively.:

$$N_i = \frac{1}{2A^e} (a_i + b_i r + c_i z) \quad (1)$$

With $2A^e = ab$, $a_i = r_j z_k - r_k z_j$, $b_i = z_j - z_k$ and $c_i = r_k - r_j$.

Consequently, given $r_1 = 0$, $r_2 = r_3 = a$, $z_1 = z_2 = 0$ and $z_3 = b$, the constant values of equation (1) are given in the table below:

i	a	b	c
1	ab	$-b$	0
2	0	b	$-a$
3	0	0	a

Table 1: Shape functions constants.

Thus, the 3 shape functions can be written as:

$$N_1 = \frac{1}{ab}(ab - br) \quad (2)$$

$$N_2 = \frac{1}{ab}(br - az) \quad (3)$$

$$N_3 = \frac{1}{ab}az \quad (4)$$

Given the relation between strains and displacements denoted by:

$$\mathbf{e} = \mathbf{D}\mathbf{u} \quad (5)$$

And in turn, the discretized version of the displacements (FEM approximation) in terms of nodal displacements is given by:

$$\mathbf{u} = \mathbf{N}\mathbf{u}^e \quad (6)$$

The matrix \mathbf{N} and vector \mathbf{u}^e are chosen to be defined respectively as:

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \quad (7)$$

$$\mathbf{u}^e = [u_{r_1} u_{z_1} u_{r_2} u_{z_2} u_{r_3} u_{z_3}]^T \quad (8)$$

Plugging equation (6) into equation (5) we have:

$$\mathbf{e} = \mathbf{D}\mathbf{N}\mathbf{u}^e = \mathbf{B}\mathbf{u}^e \quad (9)$$

Leading to the definition of the *Strain-Displacement* matrix \mathbf{B} in this context:

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_3}{r} & 0 & \frac{N_2}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \quad (10)$$

Using the definition of the shape functions for this problem, given by equations (2), (3) and (4), we obtain the final form of \mathbf{B} matrix in terms of the problem variables.

$$\mathbf{B} = \frac{1}{ab} \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{ab-br}{r} & 0 & \frac{br-az}{r} & 0 & \frac{az}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{bmatrix} \quad (11)$$

By the principle of the virtual work or variational formulation of potential energy it can be shown that the stiffness matrix of the element \mathbf{K}^e is given by:

$$\mathbf{K}^e = \int_{A^e} 2\pi r \mathbf{B}^T \mathbf{E} \mathbf{B} dA \quad (12)$$

For the triangle case the differential of area is $dA = dzdr$ and the integration limits, in order to perform the integral over the triangular area, are from 0 to a and from 0 to br/a for the variables r and z respectively.

Notice that the integration limit for the variable z was chosen to be constrained by the line which links node 1 and 3, as seen in Figure (1). Also, the 2π factor is dropped as there is not any point load.

$$\mathbf{K}^e = \int_0^a \int_0^{\frac{br}{a}} \mathbf{B}^T \mathbf{E} \mathbf{B} r dz dr \quad (13)$$

Where the elasticity matrix for $\nu = 0$ is shown in equation (14).

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (14)$$

With the \mathbf{B} matrix given in equation (11), the matrix product in equation (13) can be performed, leading to the following:

$$\mathbf{K}^e = \frac{E}{(ab)^2} \int_0^a \int_0^{\frac{br}{a}} \begin{bmatrix} rb^2 + \frac{(ab-br)^2}{r} & 0 & \frac{(ab-br)(br-az)}{r} - rb^2 & 0 & \frac{a(ab-br)z}{r} & 0 \\ 0 & \frac{b^2 r}{2} & \frac{abr}{2} & -\frac{b^2 r}{2} & -\frac{abr}{2} & 0 \\ \left(\frac{ra^2}{2} + rb^2 + \frac{(br-az)^2}{r}\right) & -\frac{1}{2}abr & \frac{az(br-az)}{r} - \frac{ar^2}{2} & \left(a^2 r + \frac{b^2 r}{2}\right) & \frac{abr}{2} & -a^2 r \\ \text{Symm.} & & \frac{z^2 a^2}{r} + \frac{ra^2}{2} & & & 0 \\ & & & & & a^2 r \end{bmatrix} dz dr$$

After the integration process of each component the stiffness matrix in terms of a and b can be found.

$$\mathbf{K}^e = E \begin{bmatrix} \frac{2b}{3} & 0 & -\frac{b}{4} & 0 & \frac{b}{12} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & -\frac{b}{6} & -\frac{a}{6} & 0 \\ -\frac{b}{4} & \frac{a}{6} & \left(\frac{a^2}{6b} + \frac{4b}{9}\right) & -\frac{a}{6} & \left(\frac{b}{18} - \frac{a^2}{6b}\right) & 0 \\ 0 & -\frac{b}{6} & -\frac{a}{6} & \left(\frac{a^2}{3b} + \frac{b}{6}\right) & \frac{a}{6} & -\frac{a^2}{3b} \\ \frac{b}{12} & -\frac{a}{6} & \left(\frac{b}{18} - \frac{a^2}{6b}\right) & \frac{a}{6} & \left(\frac{a^2}{6b} + \frac{b}{9}\right) & 0 \\ 0 & 0 & 0 & -\frac{a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix} \quad (15)$$

The software *Wolfram Mathematica* was employed to check the analytical integration of the stiffness matrix. The routine is provided in the *Appendix* section.

2 - Summation of row and columns

A careful look at the \mathbf{K}^e , given in equation (15), shows that the summation of elements in rows and columns 2,4 and 6 accounts to zero. While the summation of elements in row and columns 1,3 and 5 is a non-zero value. The rows 1,3 and 5 of the FEM model system of equations corresponds to the radial displacement DOF's, as seen in equation (8).

This result on the summation of rows and columns comes to the fact that a uniform radial displacement field no longer a *Rigid body motion* because in such case, a circumferential strain arises. For example, if a uniform radial displacement field $\mathbf{u}^e = [u_r 0 u_r 0 u_r 0]^T$ is assumed, as in Figure (2) (a), we can find the radial and circumferential strains, using equations (9) and (10).

$$\epsilon_{rr} = \frac{\partial N_2}{\partial r} u_r + \frac{\partial N_3}{\partial r} u_r + \frac{\partial N_1}{\partial r} u_r = (-b + b + 0) u_r = 0 \quad (16)$$

$$\epsilon_{\theta\theta} = \frac{N_2}{r} u_r + \frac{N_3}{r} u_r + \frac{N_1}{r} u_r = (N_1 + N_2 + N_3) \frac{u_r}{r} = \frac{u_r}{r} \quad (17)$$

As it can be seen, in contrast with a 2D plane-stress or strain case, even for a uniform radial displacement field (i.e., derivative w.r.t. to r is zero) the circumferential strain $\epsilon_{\theta\theta}$ is different than zero, leading to stresses inside the continuum model. Further, if under this displacement field a strain component is *non-zero* valued, it means that a external work is being in balance with this internal work (work done by internal stress). Thus, if this displacement field $\mathbf{u}^e = [u_r 0 u_r 0 u_r 0]^T$ is imposed in the FEM model system of equations, *non-zero* external forces must appear ($\mathbf{K}^e \mathbf{u}^e = \mathbf{f} \neq 0$), those forces are the one responsible for the *non-zero* strain-stress state in the continuum model.

In contrast, if a uniform axial displacement field is chosen such that $\mathbf{u}^e = [0 u_z 0 u_z 0 u_z]^T$, as in Figure (2) (b), due the fact that the summation of rows 2, 4 and 6 are zero, the external force vector will be zero ($\mathbf{K}^e \mathbf{u}^e = \mathbf{f} = 0$), and the *Rigid body motion* condition is achieved, as for a regular 2-D plane stress or strain problem.

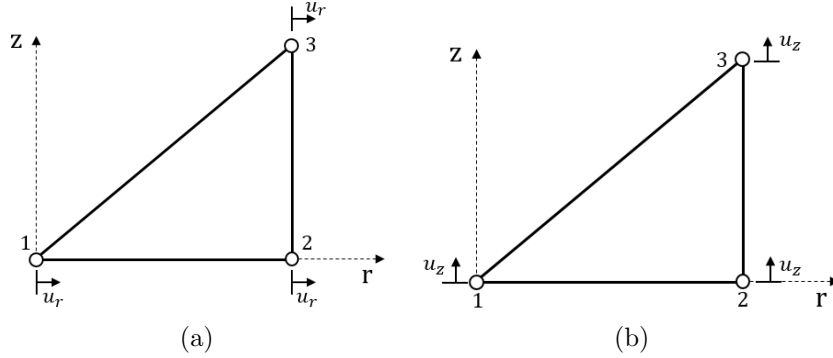


Figure 2: (a) Non Rigid Body motion;(b) Rigid Body motion

3 - Consistent nodal force vector

Given the body force vector $\mathbf{b} = [0 - \rho g]^T$, the consistent nodal force is computed as follows:

$$\mathbf{f}^e = \int_0^a \int_0^{\frac{br}{a}} \mathbf{N}^T \mathbf{b} r dz dr = \int_0^a \int_0^{\frac{br}{a}} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} r dz dr = \int_0^a \int_0^{\frac{br}{a}} -\rho g \begin{bmatrix} 0 \\ N_1 \\ 0 \\ N_2 \\ 0 \\ N_3 \end{bmatrix} r dz dr \quad (18)$$

Plugging the expression of each shape function given by equations (2),(3) and (4) we obtain:

$$\mathbf{f}^e = \frac{-\rho g}{ab} \int_0^a \int_0^{\frac{br}{a}} \begin{bmatrix} 0 \\ r(ab - br) \\ 0 \\ r(br - az) \\ 0 \\ rz \end{bmatrix} dz dr \quad (19)$$

After, performing the integration of each vector component, the consistent vector force in terms of the problem variables is found in equation (20).The software *Wolfram Mathematica* was employed to check the analytical integration of the consistent nodal force vector.The routine is provided in the *Appendix* section.

$$\mathbf{f}^e = -\frac{\rho g b a^2}{12} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{bmatrix} \quad (20)$$

It can be noticed that the two last *non-zero* force components which corresponds to the body forces in z direction in nodes 2 and 3 are higher. It is a expected value as the term $\mathbf{N}^T \mathbf{b}$ inside the integral in equation (18) is multiplied by r , giving more weight to nodes away from center line ($r = 0$). This can be physically explained by the fact that the farther from center line, for a given axisymmetric element, the more is the element volume, leading to a higher value of mass per unit of area (given a constant density ρ), and consequently higher values of body forces.

Appendix: *Wolfram Mathematica* routine to check integrals

```
B = {{-b, 0, b, 0, 0, 0}, {0, 0, 0, -a, 0, a}, {(a * b - b * r)/r, 0, (b * r - a * z)/r, 0, a * z/r, 0}, {0, -b, -a, b, a, 0}};
```

```
Es = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1/2}};
```

```
Bt = Transpose[B];
```

```
B//MatrixForm
```

```
Es//MatrixForm
```

```
Bt//MatrixForm
```

```
Int = r * Bt.Es.B;
```

```
Int//MatrixForm
```

```
Int2 = Simplify[Int];
```

```
Int2//MatrixForm;
```

```
Inx = Expand[Int2];
```

```
Inx//MatrixForm
```

```
K = (1/(a * b)^2) * Integrate[Int2, {z, 0, a}, {r, 0, (b * r)/a}];
```

```
K//MatrixForm
```

```
Ks = Simplify[K];
```

```
Ks//MatrixForm
```

```
Pv = Ks - Transpose[Ks];
```

```
Pv//MatrixForm
```

```
N1 = (a * b - b * r)/(a * b)
```

```
N2 = (b * r - a * z)/(a * b)
```

```
N3 = (a * z)/(a * b)
```

```
Nv = {{N1, 0, N2, 0, N3, 0}, {0, N1, 0, N2, 0, N3}};
```

```
Nv//MatrixForm
```

```
q = {{0}, {-g}};
```

```
q//MatrixForm
```

```
Int3 = r * Transpose[Nv].q;
```

Int3//MatrixForm

$$f = \int_0^a \int_0^{(b*r)/a} \text{Int3} dz dr;$$

f//MatrixForm