Master on Numerical Methods in Engineering

Computational Structural Mechanics and Dynamics

Assignment 4

Plane stress problem and linear triangle

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ASSIGNMENT (4): STRUCTURES OF REVOLUTION / ISOPARAMETRIC REPRES.

Assignment (4.1)

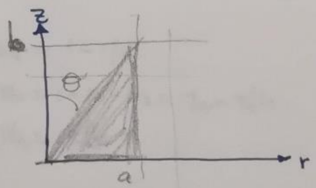
1). Compute the entries of K^e for the following axisymmetric triangle:

$r_1 = 0; \quad r_2 = r_3 = a; \quad z_1 = z_2 = 0; \quad z_3 = b$ [cylindrical coordinates (r, z, θ)]

Isotropic material with $\nu = 0$ for which the stress - strain matrix is:

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

The axisymmetric triangle:



The stiffness matrix definition of an element is:

$$K^e = \int_V B^T D B \, dV$$

Considering that the revolution problem is reduced to an axisymmetric case, the element stiffness matrix is computed as follow:

$$K_{ij}^e = 2\pi \int \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j \, r \, dr \, dz \quad \text{for any ring finite element.} \quad (1).$$

where:

$\mathbf{D} \equiv$ strain-stress matrix
$E \equiv$ Young modulus of material } Given data: $\mathbf{D} = \mathbf{E}$ as $\nu = 0$
$\mathbf{B} \equiv$

(1)

B matrix definition is:

$$\mathbf{B} = \partial N = \begin{bmatrix} \partial N_i/\partial r & 0 \\ 0 & \partial N_i/\partial z \\ N_i/r & 0 \\ \partial N_i/\partial z & \partial N_i/\partial r \end{bmatrix} \quad \text{where } i = 1,2,3 \text{ (nodes of the element)}.$$

Shape functions of the element are computed based on given data.

$$N_i = \frac{a_i + b_i r + c_i z}{2A} = \frac{1}{ab}\left(ab - br,\ br - az,\ az\right).$$

where:

$$a_i = r_j z_m - r_m z_j$$
$$b_i = z_j - z_m$$
$$c_i = r_m - r_j$$
$$A = \frac{r_i(z_j - z_k) + r_j(z_k - z_i) + r_k(z_i - z_j)}{2} = \frac{ab}{2}.$$

$$N_i \begin{cases} N_1 = 1 - r/a \\ N_2 = 1 - N_1 - N_3 = r/a - z/b. \\ N_3 = z/b \end{cases}$$

$$\mathbf{B} = \frac{1}{ab}\left[\begin{array}{cc|cc|cc} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ \frac{ab}{r} - b & 0 & b - \frac{az}{r} & 0 & \frac{az}{r} & 0 \\ 0 & -b & -a & b & a & 0 \end{array}\right] \qquad [2]$$

There are two ways to integrate Equation [1]. Numerical integration procedure is chosen for ease reasons, all quantities are going to be computed at the centroid point (r, z) where:

$$\left.\begin{aligned} r &= \frac{1}{3}\sum_{i=1}^{3} r_i = \frac{a + a + 0}{3} = \frac{2a}{3} \\ z &= \frac{1}{3}\sum_{i=1}^{3} z_i = \frac{0 + 0 + b}{3} = \frac{b}{3} \end{aligned}\right\} \text{Values to replace in } \mathbf{B} \text{ matrix}$$

Now K^e can be computed (due to numerical integration) as follow:

$$K^e = 2\pi \mathbf{B}^T \mathbf{D} \mathbf{B} r A \qquad [3]$$

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As a result:

$$K^e = 2\pi B^T D B r A = \frac{E\pi}{ab} \begin{array}{c} \begin{array}{cccccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{V} & \mathrm{VI} \end{array} \\ \left[\begin{array}{cccccc} \frac{5b^2}{4} & 0 & -\frac{3b^2}{4} & 0 & \frac{b^2}{4} & 0 \\ & \frac{b^2}{2} & \frac{ab}{2} & -\frac{b^2}{2} & -\frac{ab}{2} & 0 \\ & & \frac{5 b^2+2a^2}{4} & -\frac{ab}{2} & \frac{b^2-2a^2}{4} & 0 \\ & & & \frac{2a^2+b^2}{2} & \frac{ab}{2} & -a^2 \\ & \text{symm.} & & & \frac{b^2+2a^2}{4} & 0 \\ & & & & & a^2 \end{array}\right] \end{array} \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \\ \mathrm{V} \\ \mathrm{VI} \end{array}$$

2) Show that the sum of the rows (and columns) 2, 4, 6 of K^e must vanish and explain why. Show as well that the sum of rows (and columns) 1,3,5 do not vanish and explain why.

Take into account that Σrow p $= \Sigma$column p ; thus:

Σ Row 2 $= b^2/2 + ab/2 - ab/2 - b^2/2 = 0$

Σ Row 4 $= -b^2/2 - ab/2 + (2a^2+b^2)/2 + ab/2 - a^2 = 0.$

Σ Row 6 $= -a^2 + a^2 = 0$

Σ Row 1 $= b^2 + [(a-r)b/r]^2 - b^2 + [(a-r)(br-az)b/r^2] + [az(a-r)b/r^2] \neq 0$

Σ Row 3 $= (br-az)/r^2 \neq 0$

Σ Row 5 $= az/r^2 \neq 0.$

Rows and columns to the r-coordinate of each node are different from 0 (row 1,3,5). It is because it is not in equilibrium. Sum of displacements and force vector along r-direction is non zero.

Rows and columns 2,4,6 are in equilibrium (the sum is 0). There is equilibrium in z-direction. Sum of displacements and force vector along z-axis is 0.

(3)

3). Compute the consistent force vector f^e for gravity forces $b = \begin{bmatrix} 0 \\ -g \end{bmatrix} = \begin{bmatrix} b_r \\ b_z \end{bmatrix}$

The problem only consider body forces, thus:

$$f = -2\pi \int_S N b_r \, dr \, dz$$

Considering same procedure as in section 1), (based on N.I with centroidal coordinates), consistent force vector remains:

$$f^e = -2\pi N(r,z) \begin{bmatrix} b_r \\ b_z \end{bmatrix} rA$$

Replacing values in f^e:

$$f^e = -\frac{2\pi a}{3} \cdot \frac{ab}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

Consistent nodal force vector is:

$$f^e = +\frac{2}{9}\pi a^2 b g \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(4)

Assignment 4.1:

A 3-noded straight bar element is defined by 3 nodes: 1, 2, 3 with axial coordinates x_1, x_2, x_3.

$$\begin{cases} EA \equiv \text{axial rigidity of the element.} \\ l = x_2 - x_1 \\ u_x \equiv \text{axial displacement.} \end{cases}$$

The 3 dof are the axial node displacement u_1, u_2, u_3.

Isoparametric definition of the element:

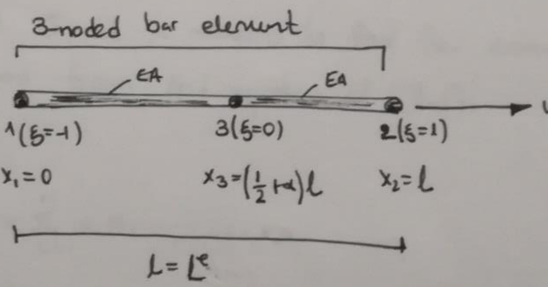
$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \qquad [7.1]$$

Shape function for a 3bar element: $N_i^e(\xi)$

Node 3 between 1 and 2 but not necessarily at $x = L/2$

For convenience, take:

$$\begin{cases} x_1 = 0 \\ x_2 = l \\ x_3 = \left(\frac{1}{2} + \alpha\right) l. \qquad -1/2 < \alpha < 1/2 \end{cases} \qquad [7.2]$$



Local coordinates.

1) From 7.2 and second eq. of 7.1 get the Jacobian $J = dx/d\xi$ in terms of l, α, ξ.

Show that:

- if $-1/4 < \alpha < 1/4 \rightarrow J > 0$ over the whole element $-1 < \xi < 1$
- if $\alpha = 0$, $\longrightarrow J = l/2$ is a constant over the element.

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3-noded bar element: It means that it is a quadratic linear element, So that, Shape functions are defined as parabolic functions for both:

- geometry interpolation
- displacement interpolation

- Shape function definition: by the general way:

$$N_j = \prod_{i=1, j\neq i}^{n} \frac{(\xi_i - \xi)}{(\xi_i - \xi_j)} \; ; \quad n \equiv \text{Number of nodes.}$$

It also has to satisfy 1st equation of [7.1]: $1 = \sum N_i \; ; \; i=1,2,3.$

Particularizing values for each node we obtain:

$$N_1 = \frac{\xi}{2}(\xi - 1)$$

$$N_2 = \frac{\xi}{2}(\xi + 1)$$

$$N_3 = 1 - \xi^2$$

In order to transform from isoparametric to cartesian coordinates, Jacobian $J = dx/d\xi$ is needed for a 1D problem.

In order to find J, it is needed to find the element geometry function that comes from 2nd equation of [7.1].

- Element geometry:

$$x = \sum_{i=1}^{n} x_i N_i \; ; \quad i = 1,2,3.$$

where:

x_i values are suggested in 7.2.

N_i shape functions for each node. Already computed.

$$x = \sum_{i=1}^{3} x_i N_i = x_1 N_1 + x_2 N_2 + x_3 N_3 = 0 + L \cdot \frac{\xi}{2}(\xi + 1) + \left(\frac{1}{2} + \alpha\right) L \cdot (1 - \xi^2)$$

- Jacobian scalar (1D):

$$J = \frac{dx}{d\xi} = \frac{L}{2} - 2\alpha L \xi \longrightarrow J = L\left(\frac{1}{2} - 2\alpha \xi\right)$$

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- Showing that $J>0$ over $-1<\xi<1$: if $-1/4<\alpha<1/4$:

$$J>0 \to \frac{1}{2}-2\alpha\xi>0.$$

→ α at boundary $\xi=-1$: $\frac{1}{2}+2\alpha>0 \to \boxed{\alpha>-1/4}$ ✓

→ α at boundary $\xi=1$: $\frac{1}{2}-2\alpha>0 \to \boxed{\alpha<1/4}$ ✓

- Showing that $J=1/2$ is a constant over the element if $\alpha=0$:

Working with node 3 (mid-node): if $\alpha=0$, it means that $x_3=\frac{1}{2}\cdot l$.
(node 3 in the middle of the element bcs.
When working with J:

$$J=\frac{1}{2}-\cancel{2\alpha\xi} \to \boxed{J=1/2 \text{ constant over the element if } \alpha=0}$$ ✓

2). Obtain the 1x3 strain displacement matrix B relating $e=du/dx=Bu^e$ where u^e is the column 3-vector of the node displacement u_1,u_2,u_3. The entries of B are functions of l,α,ξ.

$$B=\frac{dN}{dx}=J^{-1}\frac{dN}{d\xi}$$

$$J^{-1}=\left[l\left(\frac{1}{2}-2\alpha\xi\right)\right]^{-1}.$$

$$\frac{dN}{d\xi}=\left[\frac{dN_1}{d\xi} \;\Big|\; \frac{dN_2}{d\xi} \;\Big|\; \frac{dN_3}{d\xi}\right]=\left[\xi-\frac{1}{2} \;\Big|\; \xi+\frac{1}{2} \;\Big|\; -2\xi\right]$$

$$\boxed{B=\left[\frac{2\xi-1}{l(1+4\alpha)} \;\Big|\; \frac{2\xi+1}{l(1-4\alpha)} \;\Big|\; -\frac{4\xi}{l}\right]_{3\times1}}$$

(7)