"Structures of revolution"

Assignment 4.1

1. Compute the entries of $\mathbf{K}^e$ for the following axisymmetric triangle:

$$r_1 = 0, \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is,

$$\boldsymbol{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. Show that the sum of the rows (and columns) 2, 4 and 6 of $\mathbf{K}^e$ must vanish and explain why. Show as well that the sum of rows (and columns) 1, 3 and 5 does not vanish, and explain why.

3. Compute the consistent force vector $\mathbf{f}^e$ for gravity forces $\mathbf{b} = [0, -g]^T$.

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The assignment must be submitted as a pdf file named **As4-Surname.pdf** to the CIMNE virtual center.

CSMD: Assignment 4

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1 Triangle of revolution

1.1 Compute stiffness matrix

For this triangle we will consider an isoparametric interpolation $(r, z) \rightarrow (\xi, \eta)$ with shapefunctions N_i:

$$N_1 = \xi;\ N_2 = 1 - \xi - \eta;\ N_3 = \eta \tag{1}$$

With respect to the global variables (r, z), we can see that in order to maintain the interpolation so that $r(\xi, \eta) = \sum_i^3 N_i r_i$, the values of the shape-functions N_i must be:

$$N_1 = \frac{a - r}{a};\ N_2 = \frac{r}{a} - \frac{z}{b};\ N_3 = \frac{z}{b} \tag{2}$$

And then, the interpolation of the radius (with $r_1 = 0,\ r_2 = r_3 = a$) becomes

$$r = r_1 N_1 + r_2 N_2 + r_3 N_3 = a(1 - \xi - \eta) + a\eta = a(1 - \xi) \tag{3}$$

The derivatives of the shape-functions with respect to (r,z) are not dependent on any variable, only the geometry:

$$\begin{aligned} \frac{\partial N_1}{\partial r} &= -\frac{1}{a};\ \frac{\partial N_1}{\partial z} = 0 \\ \frac{\partial N_2}{\partial r} &= \frac{1}{a};\quad \frac{\partial N_2}{\partial z} = -\frac{1}{b} \\ \frac{\partial N_3}{\partial r} &= 0;\quad \frac{\partial N_3}{\partial z} = \frac{1}{b} \end{aligned} \tag{4}$$

And the relation between N_i and r:

$$\frac{N_1}{r} = \frac{\xi}{a(1 - \xi)};\ \frac{N_2}{r} = \frac{1 - \xi - \eta}{a(1 - \xi)};\ \frac{N_3}{r} = \frac{\eta}{a(1 - \xi)} \tag{5}$$

With all this expressions, we can already start building the stiffness matrix. The general expression for a revolution structure stiffness matrix calculated using the centroid rule (1 point Gauss quadrature) for the integral is:

$$\mathbf{K}^e = 2\pi w_k \mathbf{B}^T(\xi_k, \eta_k)\mathbf{E}\mathbf{B}(\xi_k, \eta_k) r(\xi_l, \eta_k) J(\xi_k \eta_k) \tag{6}$$

In this case, the integration point is the centroid of the triangle $P_k = (\frac{1}{3}, \frac{1}{3})$, with $w_k = 0.5$. The Jacobian of linear triangle transformation is twice the area of such triangle, so in this case we have that $J = 2A = ab$. The radius $r(\xi_k, \eta_k)$ at this point is, using (2), $r = \frac{2}{3}a$. **B** and **E** matrices are (after including the expressions in (4) and (5)):

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & \frac{N_2}{r} & \frac{N_3}{r} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{-1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{b} & \frac{1}{b} \\ \frac{\xi}{a(1-\xi)} & \frac{1-\xi-\eta}{a(1-\xi)} & \frac{\eta}{a(1-\xi)} & 0 & 0 & 0 \\ 0 & \frac{-1}{b} & \frac{1}{b} & \frac{-1}{a} & \frac{1}{a} & 0 \end{bmatrix}; \qquad (7)$$

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Particularizing for $P_k = (\frac{1}{3}, \frac{1}{3})$, $\mathbf{B}^{\mathrm{T}}$ **EB** becomes:

$$\mathbf{B}^T\mathbf{EB} = E \begin{bmatrix} \frac{5}{4a^2} & \frac{-3}{4a^2} & \frac{1}{4a^2} & 0 & 0 & 0 \\ & \frac{-3}{4a^2} + \frac{1}{2b^2} & \frac{1}{4a^2} + \frac{-1}{2b^2} & \frac{1}{2ba} & \frac{-1}{2ba} & 0 \\ & & \frac{1}{4a^2} + \frac{1}{2b^2} & \frac{-1}{2ba} & \frac{1}{2ba} & 0 \\ & & & \frac{1}{2a^2} & \frac{-1}{2a^2} & 0 \\ & & symm & & \frac{1}{b^2} + \frac{1}{2a^2} & \frac{-1}{b^2} \\ & & & & & \frac{1}{b^2} \end{bmatrix}$$

And $\mathbf{K}^{\mathrm{e}}$:

$$\mathbf{K}^e = 2\pi\frac{1}{3}a^2bE \begin{bmatrix} \frac{5}{4a^2} & \frac{-3}{4a^2} & \frac{1}{4a^2} & 0 & 0 & 0 \\ & \frac{-3}{4a^2} + \frac{1}{2b^2} & \frac{1}{4a^2} + \frac{-1}{2b^2} & \frac{1}{2ba} & \frac{-1}{2ba} & 0 \\ & & \frac{1}{4a^2} + \frac{1}{2b^2} & \frac{-1}{2ba} & \frac{1}{2ba} & 0 \\ & & & \frac{1}{2a^2} & \frac{-1}{2a^2} & 0 \\ & & symm & & \frac{1}{b^2} + \frac{1}{2a^2} & \frac{-1}{b^2} \\ & & & & & \frac{1}{b^2} \end{bmatrix}$$

1.2 Stiffness matrix analysis

We can see that the sum of rows (or columns) 4, 5 and 6 gives 0. These rows are related to the motion in Z direction. As we are dealing with revolution structures, the same movement in the Z direction would be just a vertical translation, and then, rigid body motion. If we look at rows 1,2 and 3 (rows linked to radial direction), this is not the case. The same motion in the radial direction would mean a deformation of the element (expansion if positive, shrinkage if negative), and thus, the relative rows of the stiffness matrix are non-zero.

1.3 Force vector for gravity forces

Following the same reasoning that was done for the stiffness matrix, the expression of the force vector is (integration done by centroid rule):

$$\mathbf{f}^e = 2\pi w_k \mathbf{N}^T(\xi_k, \eta_k)\mathbf{b}(\xi_k, \eta_k) r(\xi_k, \eta_k) J(\xi_k, \eta_k) \tag{8}$$

Matrix $\mathbf{N}$ of shape functions is (already particularized for centroid P_k, and with the same weight $w_k = 0.5$)

$$\mathbf{N} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{9}$$

The force vector is

$$\mathbf{f}^e = 2\pi \frac{-1}{9} a^2 b g \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$