Assignment 4.1

3-node straight bar element:

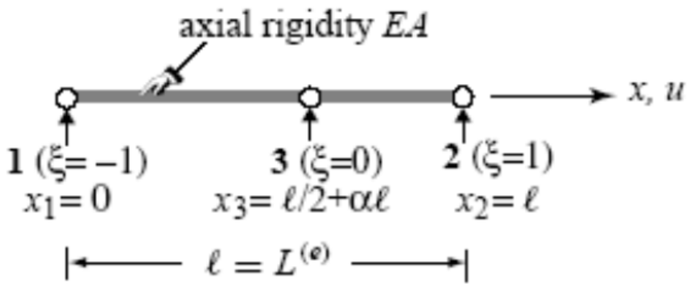


Figure.- The three-node bar element in its local system

The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

In which N^e are the shape functions of the element, with 3 degrees of freedom.

The coordinates of the nodes are displayed as follows

$$x_1 = 0 \qquad x_2 = l \qquad x_3 = \left(\frac{1}{2} + \alpha\right) l$$

Where $-\frac{1}{2} < \alpha < \frac{1}{2}$ characterizes the location of the node 3 with respect to the element canter.

1. Computation of the Jacobian matrix

The shape functions of the nodes are expressed in a quadratic form as follows

$$N_1^e = \frac{-\xi}{2}(1 - \xi)$$
$$N_2^e = \frac{\xi}{2}(\xi + 1)$$
$$N_3^e = (1 - \xi)(1 + \xi)$$

Since this is a 1D problem the Jacobian matrix is a scalar and can be computed by the following expression

$$J = \frac{dx}{d\xi} = x_1 \frac{dN_1^e}{d\xi} + x_2 \frac{dN_2^e}{d\xi} + x_3 \frac{dN_3^e}{d\xi}$$

Plugging in the previous shape functions and deriving respect to the natural coordinate the Jacobian results as

$$J = l\left(\xi + \frac{1}{2}\right) + \left(\frac{l}{2} + \alpha l\right)(-2\xi) = -2\alpha l\xi + \frac{l}{2}$$

Forcing that $J > 0$

$$\alpha\xi = \frac{1}{4}$$

For $\xi = -1$ & $\xi = 1$, the corresponding bounds of the parameter α are

$$\alpha = -\frac{1}{4} \ \& \ \alpha = \frac{1}{4}$$

Therefore,

If $-\frac{1}{4} < \alpha < \frac{1}{4}$ then J > 0 over the whole element

And if $\alpha = 0 \ \ then \ \ J = \frac{1}{2}$ is a constant over the element.

2. Computation of strain-displacement matrix B

The expression that describes the strain-displacement matrix is the following

$$\boldsymbol{B} = \frac{d\boldsymbol{N}}{dx} = J^{-}1\frac{d\boldsymbol{N}}{d\xi}$$

Where N = [N_1,N_2,N_3]

Plugging in the previous shape functions, deriving the expressions and inverting the Jacobian the B matrix results as follows

$$J^{-1} = \frac{1}{l\left(\frac{1}{2} - 2\alpha\xi\right)}$$

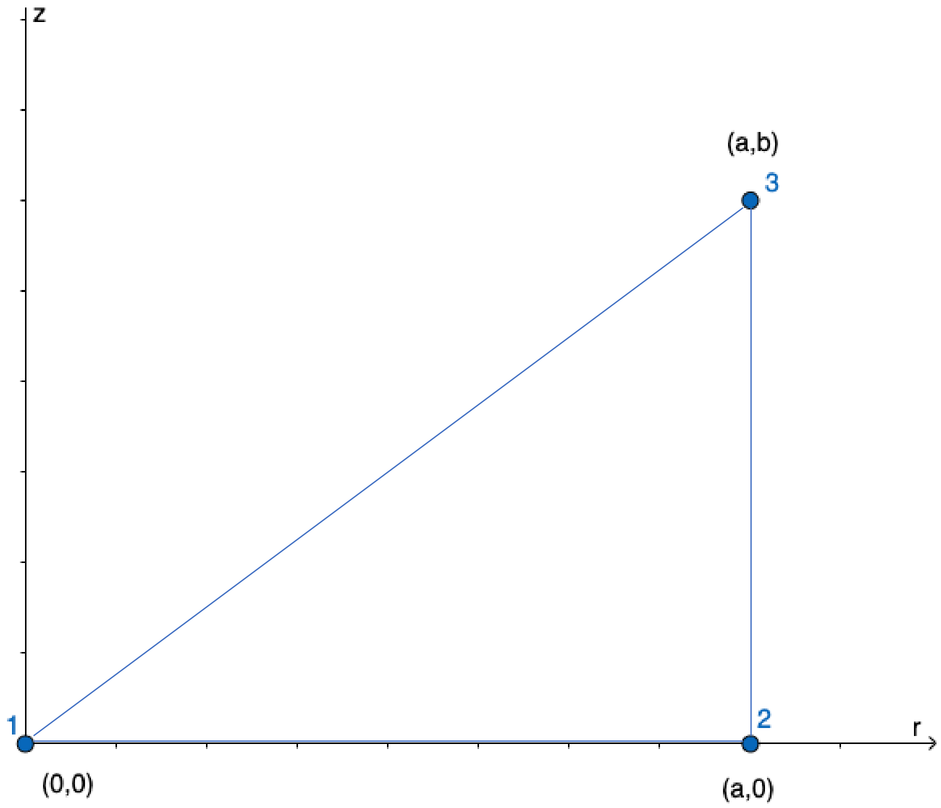
$$\frac{\mathrm{dN}}{\mathrm{d}\xi} = \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi\right]$$

$$\boldsymbol{B} = \frac{1}{l\left(\frac{1}{2} - 2\alpha\xi\right)}\begin{bmatrix} \xi - \frac{1}{2} & \xi + \frac{1}{2} & -2\xi \end{bmatrix}$$

Assignment 4.2

Axisymmetric triangle

$$r_1 = 0 \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0 \quad z_3 = b$$



Material data:

- Isotropic
- $\nu = 0$
- Stress-strain matrix:

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

1. Computation of K^e

The following expression describes the stiffness matrix

$$\boldsymbol{K^e} = \int_{V^e} \boldsymbol{B^T E B r} dr d\theta dz = 2\pi \int_{S^e} \boldsymbol{B^T E B r} dr dz$$

In order to compute the strain-displacement matrix the shape functions must be previously defined

$$\begin{cases} N_1 = 1 - \frac{r}{a} \\ N_2 = \frac{r}{a} - \frac{z}{b} \\ N_3 = \frac{z}{b} \end{cases}$$

Then, the **B** matrix is calculated as

$$D = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}$$

$$B = DN = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/b & 0 & 1/b \\ 1/r - 1/a & 0 & 1/a - z/(br) & 0 & z/(br) & 0 \\ 0 & -1/a & -1/b & 1/a & 1/b & 0 \end{bmatrix}$$

The integration limits are defined as follows

$$\boldsymbol{K^e} = 2\pi \int_0^a \int_0^{b/ar} \boldsymbol{B^T E B} r dz dr$$

Using the symbolic toolbox of Matlab to solve the integral of the expression the stiffness matrix results as

$$K^e = \frac{2\pi a^2 b}{3} E \begin{bmatrix} \frac{2b}{3} & 0 & \frac{-b}{4} & 0 & \frac{b}{12} & 0 \\ 0 & \frac{b}{6} & \frac{a}{6} & \frac{-b}{6} & \frac{-a}{6} & 0 \\ \frac{-b}{4} & \frac{a}{6} & \frac{a^2}{6b} + \frac{4b}{9} & \frac{-a}{6} & \frac{-a^2}{6b} + \frac{b}{18} & 0 \\ 0 & \frac{-b}{6} & \frac{-a}{6} & \frac{a^2}{3b} + \frac{b}{6} & \frac{a}{6} & \frac{-a^2}{3b} \\ \frac{b}{12} & \frac{-a}{6} & \frac{-a^2}{6b} + \frac{b}{18} & \frac{a}{6} & \frac{a^2}{6b} + \frac{b}{9} & 0 \\ 0 & 0 & 0 & \frac{-a^2}{3b} & 0 & \frac{a^2}{3b} \end{bmatrix}$$

2. Sum of rows and columns (2,4,6) & (1,3,5)

The sum of rows and columns (2,4,6) vanish because this axisymmetric model can withstand motion in z-direction (axis of symmetry) without generating stress with a rigid-body behaviour. However, the sum of rows and columns (1,3,5) that represents the r-direction are not null and shows that an axisymmetric model as the one analysed here cannot withstand non-symmetric movements as those in r-direction without generating stress.

3. Computation of the consistent force vector f^e for b=[0,-g]

The expression used to calculate the force vector is

$$f^e = 2\pi \int_{S^e} N^T b r dr dz$$

$$f^e = 2\pi \int_0^a \int_0^{b/ar} \begin{bmatrix} 1 - \frac{r}{a} & 0 \\ 0 & 1 - \frac{r}{a} \\ \frac{r}{a} - \frac{z}{b} & 0 \\ 0 & \frac{r}{a} - \frac{z}{b} \\ \frac{z}{b} & 0 \\ 0 & \frac{z}{b} \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} r dz dr = -\pi g a^2 b \begin{bmatrix} 0 \\ 1/12 \\ 0 \\ 3/8 \\ 0 \\ 1/8 \end{bmatrix}$$