Computational Structural Mechanics and Dynamics

As4 Isoparametric representation and Structures of revolution

Ye Mao

mao.ye@estudiant.upc.edu

Master of Numerical methods on engineering - Universitat Politècnica de Catalunya

Assignment 4.1

A 3-node straight bar element is defined by 3 nodes 1, 2 and 3 with axial coordinates x_1, x_2 and x_3 respectively as illustrated in figure below. The element has axial rigidity EA. And length $l = x_1 - x_2$, The axial displacement is $u(x)$. The 3 degrees of freedom are the axial node displacement u_1, u_2 and u_3. The isoparametric definition of element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix} \qquad (7.1)$$

In which $N_1^e(\xi)$ are the shape functions of a three bar element. Node 3 lies between 1 and 2 but is not necessarily at the midpoint $x = l/2$. For convenience define.

$$x_1 = 0, x_2 = l, x_3 = \left(\frac{1}{2} + \alpha\right) l \qquad (7.2)$$

Where $-1/2 < \alpha < 1/2$, characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between 1 and 2.

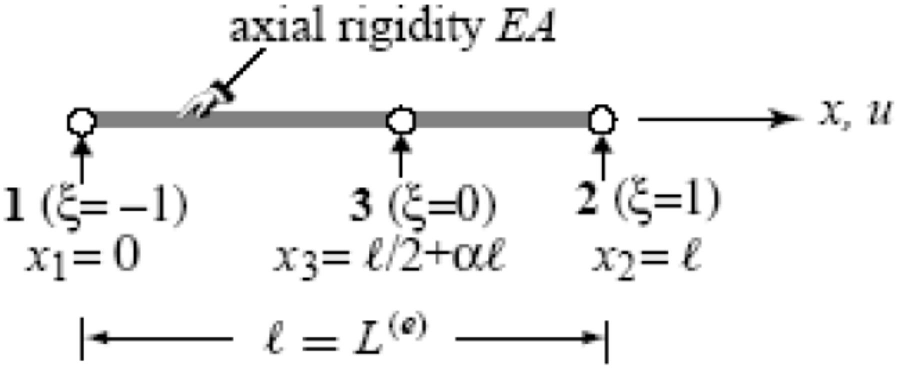

Figure.- The three-node bar element in its local system

1. **From (7.2) and the second equation of (7.1) get Jacobian $J = \frac{dx}{d\xi}$ in terms of l, α and ξ. Show that**
   If $-1/4 < \alpha < 1/4$ then $J > 0$ over the whole element $-1 < \xi < 1$
   If $\alpha = 0, J = l/2$ is a constant over the element.
   [Answer]

$$J = \frac{dx}{d\xi} = \sum_{i=1}^{3} x_i \frac{dN_i^e}{d\xi}$$

Where,

$$\xi_1 = -1, \xi_2 = 1, \xi_3 = 0$$

$$x_1 = 0, x_2 = l, x_3 = \left(\frac{1}{2} + \alpha\right) l$$

$$N_1 = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{1}{2}\xi(\xi - 1)$$

$$N_2 = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{1}{2}\xi(\xi + 1)$$

$$N_3 = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = (1 - \xi^2)$$

So,

$$J = \frac{1}{2}l(1 - 4\alpha\xi)$$

Case 1, when $-1 < \xi < 1, -1/4 < \alpha < 1/4$,
yield

$$-1/4 < \alpha\xi < 1/4$$

$$-1 < 4\alpha\xi < 1$$

$$0 < 1 - 4\alpha\xi < 2$$

Then,

$$J > 0$$

Case 2, while $\alpha = 0$,

$$J = \frac{1}{2}l$$

Which is constant.

2. **Obtain the 1×3 strain displacement matrix B relating $e = \frac{dx}{d\xi} = Bu^e$ where u^e is the column 3-vector of the node displacement u_1, u_2 and u_3. The entries of B are functions of l, α and ξ.**

   Hint: $B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi}$, where $N = [N_1, N_2, N_3]$ and J comes from item a).

   [Answer]

$$J^{-1} = \frac{1}{\frac{1}{2}l(1 - 4\alpha\xi)}$$

$$B = \frac{dN}{dx} = J^{-1}\frac{dN}{d\xi} = J^{-1}\left[\frac{dN_1}{d\xi} \quad \frac{dN_2}{d\xi} \quad \frac{dN_3}{d\xi}\right]$$

$$= \frac{1}{\frac{1}{2}l(1 - 4\alpha\xi)}\left[\frac{1}{2}(2\xi - 1) \quad \frac{1}{2}(2\xi + 1) \quad -2\xi\right]$$

$$= \left[\frac{(2\xi - 1)}{l(1 - 4\alpha\xi)} \quad \frac{(2\xi + 1)}{l(1 - 4\alpha\xi)} \quad \frac{-4\xi}{l(1 - 4\alpha\xi)}\right]$$

Assignment 4.2

1. **Compute the entries of K^e for the following axisymmetric triangle:**

$$r_1 = 0, r_2 = r_3 = a, z_1 = z_2 = 0, z_3 = b$$

The material is isotropic with $v = 0$ for which the stress-strain matrix is,

$$\boldsymbol{E} = E\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

[Answer]

We consider the unknown vector as

$$u = \begin{bmatrix} u_{r1} \\ u_{r2} \\ u_{r3} \\ u_{z1} \\ u_{z2} \\ u_{z3} \end{bmatrix}$$

In case of using a different vector of unknowns, the order of rows and columns of the stiffness matrix have to be permutated.

Firstly, the shape functions

$$N_1^e = \xi, N_2^e = \eta, N_3^e = 1 - (\xi + \eta)$$

Linear relation

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

Now, calculate Jacobian

$$J = \frac{\partial(r,z)}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \sum_{i=1}^{3} \begin{bmatrix} r_i \frac{\partial N_i^e}{\partial \xi} & z_i \frac{\partial N_i^e}{\partial \xi} \\ r_i \frac{\partial N_i^e}{\partial \eta} & z_i \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix}$$

$$J^{-1} = \frac{\partial(\xi,\eta)}{\partial(r,z)} = \frac{1}{ab}\begin{bmatrix} -b & b \\ 0 & -a \end{bmatrix}$$

Then train vector

$$\boldsymbol{B} = \boldsymbol{DN} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1^e & N_2^e & N_3^e & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1^e & N_2^e & N_3^e \end{bmatrix} = \begin{bmatrix} \boldsymbol{q_r} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{q_z} \\ \boldsymbol{q_\theta} & \boldsymbol{0} \\ \boldsymbol{q_z} & \boldsymbol{q_r} \end{bmatrix}$$

$$\boldsymbol{q_r} = \begin{bmatrix} \frac{\partial N_1^e}{\partial r} & \frac{\partial N_2^e}{\partial r} & \frac{\partial N_3^e}{\partial r} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial r} & \frac{\partial N_2^e}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_2^e}{\partial \eta}\frac{\partial \eta}{\partial r} & \frac{\partial N_3^e}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial N_3^e}{\partial \eta}\frac{\partial \eta}{\partial r} \end{bmatrix}$$

$$= \left[-\frac{1}{a} \quad \frac{1}{a} \quad 0\right]$$

$$\boldsymbol{q_z} = \left[\frac{\partial N_1^e}{\partial z} \quad \frac{\partial N_2^e}{\partial z} \quad \frac{\partial N_3^e}{\partial z}\right]$$

$$= \left[\frac{\partial N_1^e}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_1^e}{\partial \eta}\frac{\partial \eta}{\partial z} \quad \frac{\partial N_2^e}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_2^e}{\partial \eta}\frac{\partial \eta}{\partial z} \quad \frac{\partial N_3^e}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_3^e}{\partial \eta}\frac{\partial \eta}{\partial z}\right]$$

$$= \left[0 \quad -\frac{1}{b} \quad \frac{1}{b}\right]$$

$$\boldsymbol{q_\theta} = \left[\frac{N_1^e}{r} \quad \frac{N_2^e}{r} \quad \frac{N_3^e}{r}\right] = \left(\sum_{i=1}^{3} r_i N_i^e\right)^{-1} [N_1^e \quad N_2^e \quad N_3^e]$$

$$= \frac{1}{a(1-\xi)}[\xi \quad \eta \quad 1-(\xi+\eta)]$$

Finally, the element stiffness computed as

$$\boldsymbol{K^e} = \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}-\xi} \boldsymbol{G}(\xi,\eta) d\eta d\xi$$

Where

$$\boldsymbol{G}(\xi,\eta) = \boldsymbol{B^T}(\xi,\eta)\boldsymbol{E}\boldsymbol{B}(\xi,\eta)\boldsymbol{r}\big((\xi,\eta)\big)\boldsymbol{J}(\xi,\eta)$$

$$= \begin{bmatrix} \boldsymbol{q_r^T} & \mathbf{0} & \boldsymbol{q_\theta^T} & \boldsymbol{q_z^T} \\ \mathbf{0} & \boldsymbol{q_z^T} & \mathbf{0} & \boldsymbol{q_r^T} \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{q_r} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{q_z} \\ \boldsymbol{q_\theta} & \mathbf{0} \\ \boldsymbol{q_z} & \boldsymbol{q_r} \end{bmatrix} \left(\sum_{i=1}^{3} r_i N_i^e\right) \left|\begin{vmatrix} -a & -b \\ 0 & -b \end{vmatrix}\right|$$

$$= E \begin{bmatrix} \boldsymbol{q_r^T} & \mathbf{0} & \boldsymbol{q_\theta^T} & \boldsymbol{q_z^T} \\ \mathbf{0} & \boldsymbol{q_z^T} & \mathbf{0} & \boldsymbol{q_r^T} \end{bmatrix} \begin{bmatrix} \boldsymbol{q_r} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{q_z} \\ \boldsymbol{q_\theta} & \mathbf{0} \\ \frac{1}{2}\boldsymbol{q_z} & \frac{1}{2}\boldsymbol{q_r} \end{bmatrix} a(1-\xi)ab$$

$$= Ea^2b(1-\xi) \begin{bmatrix} \boldsymbol{q_r^T q_r} + \boldsymbol{q_\theta^T q_\theta} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{q_z^T q_z} & \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{q_z^T q_r} \\ \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{q_r^T q_z} & \boldsymbol{q_z^T q_z} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{q_r^T q_r} \end{bmatrix}$$

Where,

$$\boldsymbol{q_r^T q_r} = \begin{bmatrix} -\frac{1}{a} \\ \frac{1}{a} \\ 0 \end{bmatrix} \left[-\frac{1}{a} \quad \frac{1}{a} \quad 0\right] = \begin{bmatrix} \frac{1}{a^2} & -\frac{1}{a^2} & 0 \\ -\frac{1}{a^2} & \frac{1}{a^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{q_\theta^T q_\theta} = \frac{1}{a^2(1-\xi)^2} \begin{bmatrix} \xi \\ \eta \\ 1-(\xi+\eta) \end{bmatrix} [\xi \quad \eta \quad 1-(\xi+\eta)]$$

$$= \frac{1}{a^2(1-\xi)^2} \begin{bmatrix} \xi^2 & \xi\eta & \xi(1-(\xi+\eta)) \\ \xi\eta & \eta^2 & \eta(1-(\xi+\eta)) \\ \xi(1-(\xi+\eta)) & \eta(1-(\xi+\eta)) & (1-(\xi+\eta))(1-(\xi+\eta)) \end{bmatrix}$$

$$
\boldsymbol{q_z^T q_z} = \begin{bmatrix} 0 \\ -\frac{1}{b} \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{b} & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{b^2} & -\frac{1}{b^2} \\ 0 & -\frac{1}{b^2} & \frac{1}{b^2} \end{bmatrix}
$$

$$
\boldsymbol{q_r^T q_z} = \begin{bmatrix} -\frac{1}{a} \\ \frac{1}{a} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{b} & \frac{1}{b} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{ab} & -\frac{1}{ab} \\ 0 & -\frac{1}{ab} & \frac{1}{ab} \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\boldsymbol{q_z^T q_r} = \begin{bmatrix} 0 \\ -\frac{1}{b} \\ \frac{1}{b} \end{bmatrix} \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{ab} & -\frac{1}{ab} & 0 \\ -\frac{1}{ab} & \frac{1}{ab} & 0 \end{bmatrix}
$$

$$
\boldsymbol{G}(\xi,\eta) = Ea^2b(1-\xi)\cdot
$$

$$
\begin{bmatrix}
\frac{1}{a^2}+\frac{\xi^2}{a^2(1-\xi)^2} & 0 & -\frac{1}{a^2}+\frac{\xi\eta}{a^2(1-\xi)^2} & 0 & \frac{\xi(1-(\xi+\eta))}{a^2(1-\xi)^2}-\frac{1}{2}\frac{1}{b^2} & 0 \\
0 & \frac{1}{2}\frac{1}{a^2} & \frac{1}{2}\frac{1}{ab} & -\frac{1}{2}\frac{1}{a^2} & -\frac{1}{2}\frac{1}{ab} & 0 \\
-\frac{1}{a^2}+\frac{\xi\eta}{a^2(1-\xi)^2} & \frac{1}{2}\frac{1}{ab} & \frac{1}{a^2}+\frac{\eta^2}{a^2(1-\xi)^2} & -\frac{1}{2}\frac{1}{ab} & \frac{\eta(1-(\xi+\eta))}{a^2(1-\xi)^2} & 0 \\
0 & -\frac{1}{2}\frac{1}{a^2} & -\frac{1}{2}\frac{1}{ab} & \frac{1}{b^2}+\frac{1}{2}\frac{1}{a^2} & \frac{1}{2}\frac{1}{ab} & -\frac{1}{b^2} \\
\frac{\xi(1-(\xi+\eta))}{a^2(1-\xi)^2} & -\frac{1}{2}\frac{1}{ab} & \frac{\eta(1-(\xi+\eta))}{a^2(1-\xi)^2}-\frac{1}{2}\frac{1}{b^2} & \frac{1}{2}\frac{1}{ab} & \frac{(1-(\xi+\eta))(1-(\xi+\eta))}{a^2(1-\xi)^2}+\frac{1}{2}\frac{1}{b^2} & 0 \\
0 & 0 & 0 & -\frac{1}{b^2} & 0 & \frac{1}{b^2}
\end{bmatrix}
$$

2. **Show that the sum of the rows (and columns) 2,4,6 of K^e mush vanish and explain why.**
   Show as well that the sum of rows (and columns)1,3 and 5 does not vanish, and explain why.
   [Answer]
   The differential from the stiffness matrix was calculated in the previous point. The sum of the rows (and columns) 2 4 and 6 vanish while the sum of the rows (and columns)1 3 and 5 does not vanish.
   Obviously, the sum of vectors q_r and q_z vanish. Because we define the shape function that their sum is 1 over all the points of the element. So, their derivatives are 0.
   The rows 2 4 and 6 only contain tensorial product of q_r and q_z. So, the sum is null.
   The rows 1 3 and 5 contain term of tensorial product of q_θ which is composed of shape function divided by radius. All the terms are positive over all the element, and their sum will be positive over the whole element. So, it cannot vanish after

integration.

We apply the same reason for the columns since the matrix is symmetric.

The physical reason of the phenomenon is that the sum of all the columns represent the nodal force at each degree of freedom for unitary nodal displacements. In the z coordinate, this is translated as a rigid body motion. So, it does not produce any reaction. However, in the case of the r coordinate it is translated to a training of the material in the azimuthal direction which is represent in the q_θ with the radius of circumference increasing. That is the reason there is a reaction against the action.

3. Compute the consistent force vector $\boldsymbol{f^e}$ for gravity forces $\boldsymbol{b} = [\boldsymbol{0}, -\boldsymbol{g}]^{\boldsymbol{T}}$

[Answer]

$$\boldsymbol{f^e} = \int_{\boldsymbol{0}}^{\boldsymbol{1}} \int_{\boldsymbol{0}}^{\boldsymbol{1-\xi}} \boldsymbol{N^T b r J} d\eta d\xi$$

$$= \int_{\boldsymbol{0}}^{\boldsymbol{1}} \int_{\boldsymbol{0}}^{\boldsymbol{1-\xi}} \begin{bmatrix} \xi & 0 \\ \eta & 0 \\ 1-(\xi+\eta) & 0 \\ 0 & \eta \\ 0 & \eta \\ 0 & 1-(\xi+\eta) \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} \left(\sum_{i=1}^{3} r_i N_i^e \right) \left| \begin{bmatrix} -a & -b \\ 0 & -b \end{bmatrix} \right| d\eta d\xi$$

$$= a^2 b g \int_{\boldsymbol{0}}^{\boldsymbol{1}} \int_{\boldsymbol{0}}^{\boldsymbol{1-\xi}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\xi \\ -\eta \\ -1+(\xi+\eta) \end{bmatrix} (1-\xi) d\eta d\xi$$

$$= a^2 b g \int_{\boldsymbol{0}}^{\boldsymbol{1}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\xi(1-\xi)^2 \\ -\dfrac{(1-\xi)^3}{2} \\ -(1-\xi)^3 + \dfrac{(1-\xi)^3}{2} \end{bmatrix} d\xi$$

$$= a^2 b g \int_{\boldsymbol{0}}^{\boldsymbol{1}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\xi(1-\xi)^2 \\ -\dfrac{(1-\xi)^3}{2} \\ -\dfrac{(1-\xi)^3}{2} \end{bmatrix} d\xi$$

$$= a^2 bg \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\xi(1-\xi)^2 \\ -\frac{(1-\xi)^3}{2} \\ -\frac{(1-\xi)^3}{2} \end{bmatrix}$$

$$= a^2 bg \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{12} \\ -\frac{1}{8} \\ -\frac{1}{8} \end{bmatrix}$$