UPC - BARCELONA TECH
MSc Computational Mechanics
Spring 2018

Computational Solid Mechanics & Dynamics

Assignment 5

Due 12/03/2018

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"Isoparametric representation"

Problem 5.1

Consider a three-node bar element referred to the natural coordinate ξ. The two end nodes and the mid node are identified as 1, 2 and 3 respectively. The natural coordinates of nodes 1, 2 and 3 are $\xi = -1$, $\xi = 1$ and $\xi = 0$, respectively. The variation of the shape functions $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ is sketched in the figure below. These functions must be quadratic polynomials in ξ:

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2 \quad N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2 \quad N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

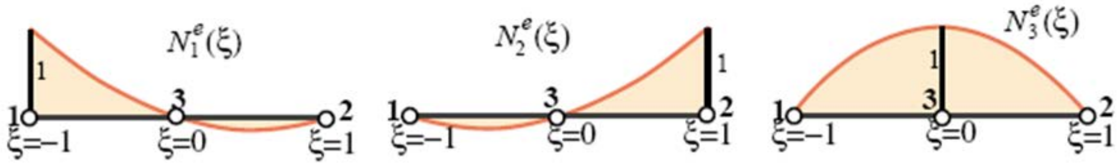


Figure.- Isoparametric shape functions for 3-node bar element (sketch).
Node 3 has been drawn at the 1-2 midpoint but it may be moved away from it.

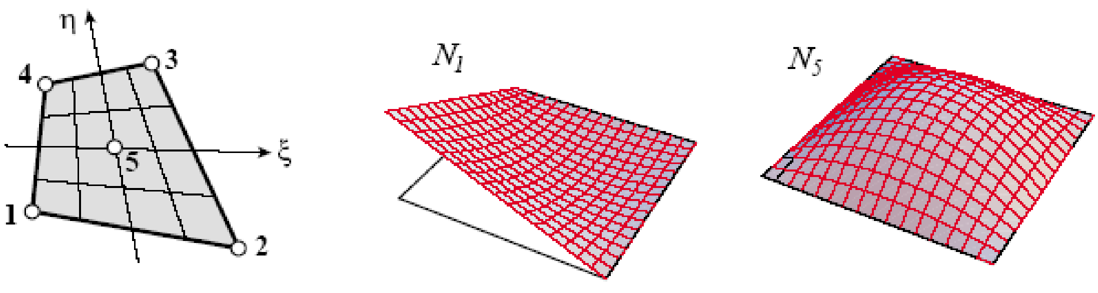
a) Determine the coefficients $a_0,\ldots,c_2$ using the node value conditions depicted in figure. For exemple $N^e_1 = 1$ for ξ=1 and 0 for the rest of natural coordinates. The rest of the nodes follow the same scheme.

b) Verify that their sum is identically one.

c) Calculate their derivatives respect to the natural coordinates.

Problem 5.2

A five node quadrilateral element has the nodal configuration shown if the figure with two perspective views of N^e_1 and N^e_5. Find five shape functions N^e_i, i=1,…,5 that satisfy compatibility and also verify that their sum is unity.



Hint: develop $N_5(\xi,\eta)$ first for the 5-node quad using the line-product method. Then the corner shape functions $\underline{N}_i(\xi,\eta)$, i=1,2,3,4, for the 4-node quad (already given in the notes). Finally combine $N_i = \underline{N}_i + \alpha N_5$ determining α so that all N_i vanish node 5. Check that $N_1 + N_2 + N_3 + N_4 + N_5 = 1$ identically.

"Convergence rerquirements"

Problem 5.3

Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for these elements:

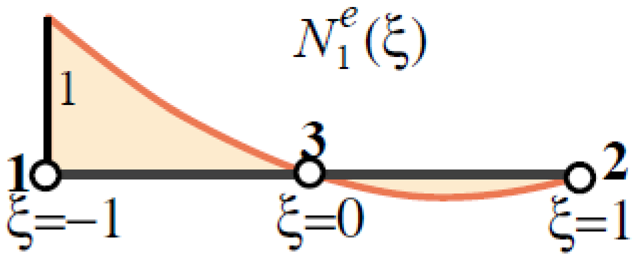
1. the 8-node hexahedron
2. the 20-node hexahedron
3. the 27-node hexahedron
4. the 64-node hexahedron

Date of Assignment: 5 / 03 / 2018
Date of Submission: 12 / 03 / 2018

The assignment must be submitted as a pdf file named **As5-Surname.pdf** to the CIMNE virtual center.

5.1 Consider a three node bar element to natural coordinate ζ. The natural coordinate of nodes 1,2 and 3 are $\zeta = -1, \zeta = 0$ and $\zeta = 1$ respectively. These functions must be quadratic polynomials in ζ

a) For Node 1

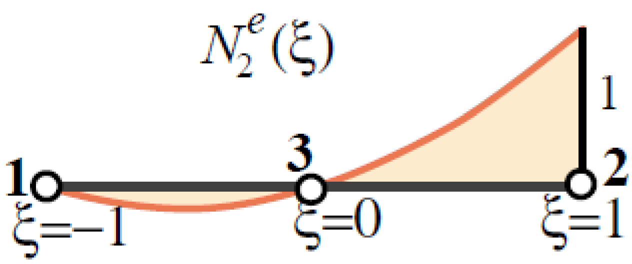


$$N_1^e = a_0 + a_1\zeta + a_2\zeta^2$$

From the above figure it can be seen that the values for the shape function for node 1 are
$N_1^e(\zeta = -1) = 1$
$N_1^e(\zeta = 0) = 0$
$N_1^e(\zeta = 1) = 0$
If we use the values of node values at the natural coordinates we can find out values for constants a_0, a_1 and a_2
$N_1^e(\zeta = -1) = 1 = a_0 - a_1 + a_2$
$N_1^e(\zeta = 0) = 0 = a_0$
$N_1^e(\zeta = 1) = 0 = a_0 + a_1 + a_2$
Hence, the values for constants are as follows
$a_0 = 0, a_1 = \frac{-1}{2}, a_2 = \frac{1}{2}$ Therefore the shape function for node 1 is

$$\boxed{N_1^e = \frac{-\zeta}{2} + \frac{\zeta^2}{2}}$$

For Node 2

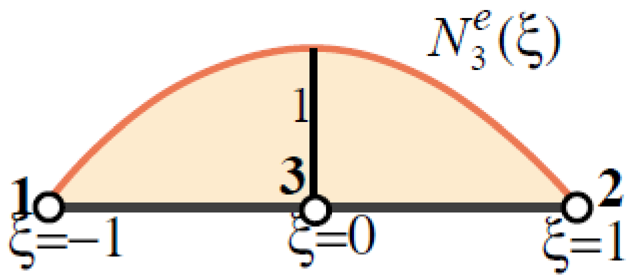


$$N_2^e = b_0 + b_1\zeta + b_2\zeta^2$$

From the above figure it can be seen that the values for the shape function for node 1 are
$N_2^e(\zeta = -1) = 0$
$N_2^e(\zeta = 0) = 0$
$N_2^e(\zeta = 1) = 1$
If we use the values of node values at the natural coordinates we can find out values for constants b_0, b_1 and b_2
$N_2^e(\zeta = -1) = 0 = b_0 - b_1 + b_2$
$N_2^e(\zeta = 0) = 1 = b_0$
$N_2^e(\zeta = 1) = 0 = b_0 + b_1 + b_2$
Hence, the values for constants are as follows
$b_0 = 0, b_1 = b_2 = \frac{1}{2}$
Therefore the shape function for node 2 is

$$\boxed{N_2^e = \frac{\zeta}{2} + \frac{\zeta^2}{2}}$$

For Node 3



$$N_3^e = c_0 + c_1\zeta + c_2\zeta^2$$

From the above figure it can be seen that the values for the shape function for node 1 are
$N_3^e(\zeta = -1) = 0$
$N_3^e(\zeta = 0) = 1$
$N_3^e(\zeta = 1) = 0$
If we use the values of node values at the natural coordinates we can find out values for constants c_0, c_1 and c_2
$N_3^e(\zeta = -1) = 0 = c_0 - c_1 + c_2$
$N_3^e(\zeta = 0) = 1 = c_0$
$N_3^e(\zeta = 1) = 0 = c_0 + c_1 + c_2$
Therefore the shape function for node 3 is

$$\boxed{N_3^e = 1 - \zeta^2}$$

b) Verify the sum of shape functions is one

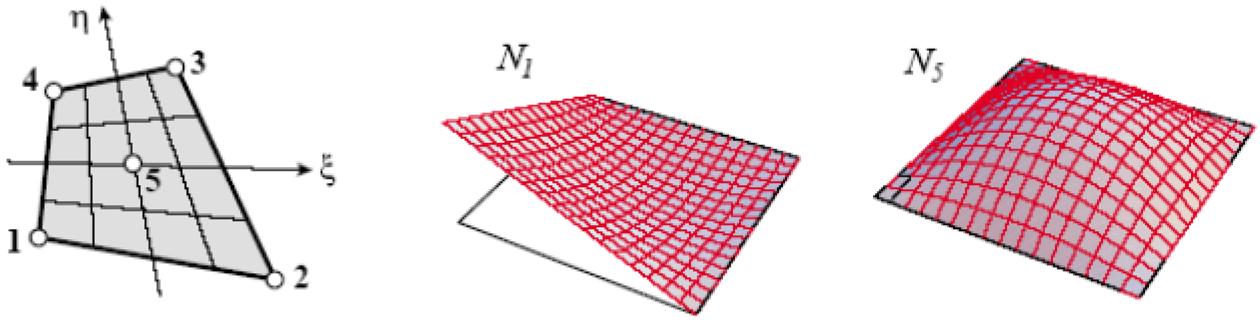
$$N_1^e + N_2^e + N_3^e = (\frac{-\zeta}{2} + \frac{\zeta^2}{2}) + (\frac{\zeta}{2} + \frac{\zeta^2}{2}) + (1 - \zeta^2) = 1$$

Hence it is verified that the sun of shape functions is 1. Thus proving that the shape functions satisfy compatibility equation

c) The derivatives with respect t natural coordinates

$$\frac{\partial N_1^e}{\partial \zeta} = \frac{-1}{2} + \zeta$$
$$\frac{\partial N_2^e}{\partial \zeta} = \frac{1}{2} + \zeta$$
$$\frac{\partial N_3^e}{\partial \zeta} = -2\zeta$$

5.2 A five node quadrilateral element has the nodal configuration shown in the figure. To find shape functions for the same



From the lecture notes, we already have the shape functions for the four corner nodes. Which are as follows

$$\underline{N}_1^e = \tfrac{1}{4}(1-\zeta)(1-\eta)$$
$$\underline{\mathrm{N}}_2^e = \tfrac{1}{4}(1+\zeta)(1-\eta)$$
$$\underline{\mathrm{N}}_3^e = \tfrac{1}{4}(1+\zeta)(1+\eta)$$
$$\underline{\mathrm{N}}_4^e = \tfrac{1}{4}(1-\zeta)(1+\eta)$$

We calculate the shape function for node 5 as follows
$N_5(\zeta,\eta) = c_5(1-\zeta)(1-\eta)(1+\zeta)(1+\eta) = c_5(1-\zeta^2)(1-\eta^2)$
From the figure it is clear that the value of N_5 at various natural coordinates
$N_5(0,0) = c_5(1-0^2)(1-0^2)$
$\therefore c_5 = 1$

$$\therefore \boxed{N_5 = (1-\zeta^2)(1-\eta^2)}$$

It can be seen from the figure that for 4 node quadrilateral that shape function for nodes $1,2,3,4$ does not become zero at node 5 . Hence we combine $N_i = \underline{N}_i + \alpha N_5$, determining α so that all N_i vanish node 5. This simply means that we modify the shape functions of 4 node quadrilateral so that shape functions for respective nodes $(1,2,3,4)$ become zero at natural coordinates (0,0). $N_i = \underline{N_i} + \alpha N_5$
$N_1 = \underline{N_1} + \alpha N_5 = \frac{1}{4}(1-\zeta)(1-\eta) + \alpha(1-\zeta^2)(1-\eta^2)$
For node 1, the value of N_1 is zero at node 5 $(0,0)$.
$0 = \frac{1}{4}(1-0)(1-0) + \alpha(1-0^2)(1-0^2)$

$\therefore \alpha = -\dfrac{-1}{4}$ From the above equation and repeating this calculation for each node it is found out that the value of alpha for each node shape function is $\dfrac{-1}{4}$. Therefore the new shape functions for 5 node quadrilateral are as follows

$$N_1 = \tfrac{1}{4}(1-\zeta)(1-\eta) - \frac{1}{4}(1-\zeta^2)(1-\eta^2)$$
$$\mathrm{N}_2 = \tfrac{1}{4}(1+\zeta)(1-\eta) - \frac{1}{4}(1-\zeta^2)(1-\eta^2)$$
$$\mathrm{N}_3 = \tfrac{1}{4}(1+\zeta)(1+\eta) - \frac{1}{4}(1-\zeta^2)(1-\eta^2)$$
$$\mathrm{N}_4 = \tfrac{1}{4}(1-\zeta)(1+\eta) - \frac{1}{4}(1-\zeta^2)(1-\eta^2)$$
$$\mathrm{N}_5 = (1-\zeta^2)(1-\eta^2)$$

5.3 Finding minimum integration rules of Gauss Product type type gives a rank sufficient matrix for following elements

To solve this problem we are going to consider following terms
n_F = number of element degrees of freedom
n_R = number of independent rigid body modes
r = rank of stiffness matrix K^e
n_E = order of stress strain matrix **E**.
n_G = minimum number of Gauss Points
We know that for an element to be rank sufficient, r = n_F - n_R. And furthermore if the we want to numerically integrate, and to attain rank sufficiency, the product $n_E n_G \geq n_F - n_R$
Using all the above condition, the minimum Gauss point, and number of Gauss points actual used are calculated
1) 8 node hexahedron

$n_F = 8 * 3 = 24$
$n_R = 6$
$n_E = 6$
$n_G = ?$
r = n_F - n_R $= 24 - 6 = 18$
n_E $n_G \geq n_F$ - n_R
$6.n_G \geq 18$
The minimum $n_G = 3$
The minimum Gauss integration rule to be used for rank sufficiency is $2 \times 2 \times 2$
2) 20 node hexahedron

$n_F = 20 * 3 = 60$
$n_R = 6$
$n_E = 6$
$n_G = ?$
r = n_F - n_R $= 60 - 6 = 54$
$n_E n_G \geq n_F - n_R$
$6.n_G \geq 54$
The minimum $n_G = 9$
The minimum Gauss integration rule to be used for rank sufficiency is $3 \times 3 \times 3$

3) 27 node hexahedron

$n_F = 27 * 3 = 81$

$n_R = 6$

$n_E = 6$

$n_G = ?$

r $= n_F$ - $n_R = 81 - 6 = 75$

$n_E n_G \geq n_F - n_R$

$6.n_G \geq 75$ The minimum $n_G = 12.5$

The minimum Gauss integration rule to be used for rank sufficiency is $3 \times 3 \times 3$

4) 64 node hexahedron

$n_F = 64 * 3 = 192$

$n_R = 6$

$n_E = 6$

$n_G = ?$

r $= n_F$ - $n_R = 192 - 6 = 186$

$n_E n_G \geq n_F - n_R$

$6.n_G \geq 186$ The minimum $n_G = 31$

The minimum Gauss integration rule to be used for rank sufficiency is $4 \times 4 \times 4$