Assignment 5

Computational Structural Mechanics and Dynamics

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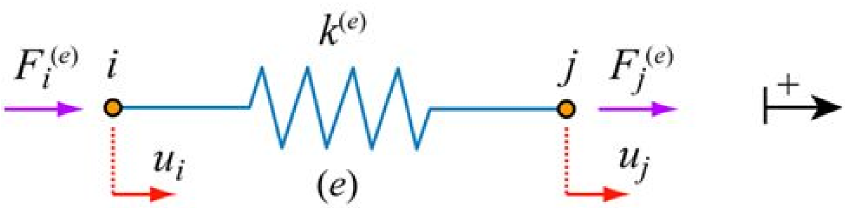


Contents

On "Convergence requirements"

1 Assignment 5.1

The isoparametric definition of the straight–node bar element in its local system x is,

$$\begin{bmatrix} 1 \\ \bar{u} \\ \bar{x} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

Here ξ is the isoparametric coordinate that takes the values –1, 1 and 0 at nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element.

For simplicity, take $\bar{x}_1 = 0$, $\bar{x}_2 = \frac{l}{2} + \alpha l$ and $\bar{x}_1 = l$. Here l is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{l}{2}$.

Show that the minimum α (minimal in absolute value sense) for which $J = \frac{d\bar{x}}{d\xi}$ vanishes at a point in the element are $\pm\frac{1}{4}$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinite at an end point.

1.1 Solution

For a quadratic Lagrange element with three nodes at $\xi_1 = -1$, $\xi_2 = 0$ and $\xi = 1$ the shape functions are:

$$N_1 = \frac{1}{2}\xi(\xi - 1)$$

$$N_2 = (1 - \xi^2)$$

$$N_3 = \frac{1}{2}\xi(\xi + 1)$$

A parametric interpolation of the element geometry yields as follows:

$$x = \sum_{i=1}^{3} N_i(\xi)x_i$$

Substituting the shape functions and the values of x_i:

$$x = \frac{1}{2}\xi(\xi - 1)(0) + (1 - \xi^2)(\frac{1}{2} + \alpha)l + \frac{1}{2}\xi(\xi + 1)l$$

$$x = -\frac{l}{2}[2\xi^2\alpha - \xi - 2\alpha - 1]$$

Where the Jacobian is defined as:

$$J = \frac{dx}{d\xi} = -\frac{l}{2}[4\xi\alpha - 1]$$

We know that J will vanish when $J = 0$, therefore:

$$J = \frac{dx}{d\xi} = -\frac{l}{2}[4\xi\alpha - 1] = 0$$

Solving for alpha:

$$\alpha = \frac{1}{4\xi}$$

For the cases of $\xi = 1$ and $\xi = -1$:

$$\alpha = |\frac{1}{4}| \qquad <--- \ J \ will \ vanish$$

This result is a singularity because in the case that α tends to $|\frac{1}{4}$ the inverse of the Jacobian will tend to infinite.

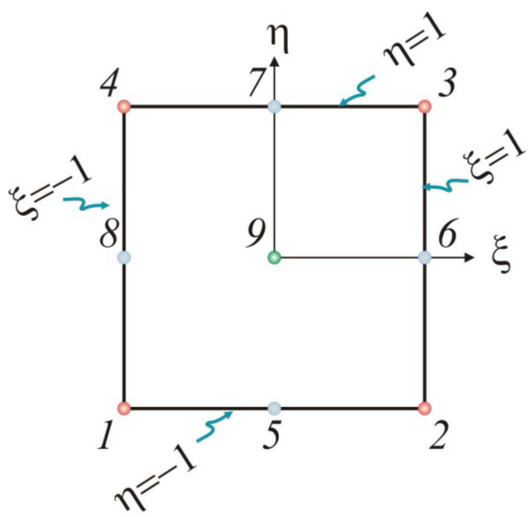
2 Assignment 5.2

Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5,6,7,8 are at the midpoint of the sides 1–2, 2–3, 3–4 and 4–1, respectively, and 9 at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.

2.1 Solution

The element described in the problem is the following:



The shape functions that describe this element are the following:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta$$

$$N_2 = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta$$

$$N_4 = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta$$

$$N_5 = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta$$
$$N_6 = \frac{1}{2}(1+\xi)*(1-\eta^2)\xi$$
$$N_7 = \frac{1}{2}(1-\xi^2)*(1+\eta)\eta$$
$$N_8 = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi$$
$$N_9 = (1-\xi^2)(1-\eta^2)$$

The approximation of x and y are obtained as follows:

$$x = \sum_{i=1}^{3} N_i(\xi,\eta)x_i$$

$$y = \sum_{i=1}^{3} N_i(\xi,\eta)y_i$$

After algebraic simplification, the approximation yields:

$$x = \frac{L(\xi+1)(\alpha\eta^2 - \alpha\eta - \alpha\xi\eta^2 + \alpha\xi\eta + 1)}{2}$$
$$y = -\frac{L(\eta+1)(\eta - \xi^2\eta + \xi^2 - 2)}{2}$$

The next step is to compute the Jacobian, which is defined as:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Where:

$$\frac{\partial x}{\partial \xi} = \frac{L(-2\alpha\xi\eta^2 + 2\alpha\xi\eta + 1)}{2}$$
$$\frac{\partial y}{\partial \xi} = L\xi(\eta^2 - 1)$$
$$\frac{\partial x}{\partial \eta} = -\frac{L\alpha(\xi^2-1)(2\eta-1)}{2}$$
$$\frac{\partial y}{\partial \eta} = \frac{L(2\eta\xi^2 - 2\eta + 1)}{2}$$

Substituting the values, we obtain the expression for the determinant of the Jacobian as follows:

$$|J| = det(\begin{bmatrix} \frac{L(-2\alpha\xi\eta^2+2\alpha\xi\eta+1)}{2} & L\xi(\eta^2-1) \\ -\frac{L\alpha(\xi^2-1)(2\eta-1)}{2} & \frac{L(2\eta\xi^2-2\eta+1)}{2} \end{bmatrix})$$
$$|J| = \frac{L^2(2\alpha\xi^3\eta^2 - 4\alpha\xi^3\eta + 2\alpha\xi^3 + 2\xi^2\eta - 4\alpha\xi\eta^2 + 6\alpha\xi\eta - 2\alpha\xi - 2\eta + 1)}{4}$$

We are asked to move node 5 tangentially towards node 2, therefore the values for node 2 of ξ and η are:

$$\xi = 1 \qquad \eta = -1$$

Therefore:

$$|J| = -\frac{(L^2 * (4 * a - 1)}{4}$$

To know the value where the determinant of the Jacobian vanishes, we equal $|J|$ to zero and solve for α:

$$\alpha = \frac{1}{4}$$

3 Discussion

Once assignments 5.1 and 5.2 have been carried out, it can be concluded from both that the node cannot move equal to or beyond half the distance between the central node (5) and the corner (2) since the determinant of the Jacobian will be undetermined. And for a case in which the Jacobian determinant is less than 0, it will represent a negative area (physically impossible), in addition to contributing negative stiffness values to our stiffness matrix (physically impossible). So it will represent generating a higher-order mesh or refining the same mesh to represent that geometry without indeterminacy problems.