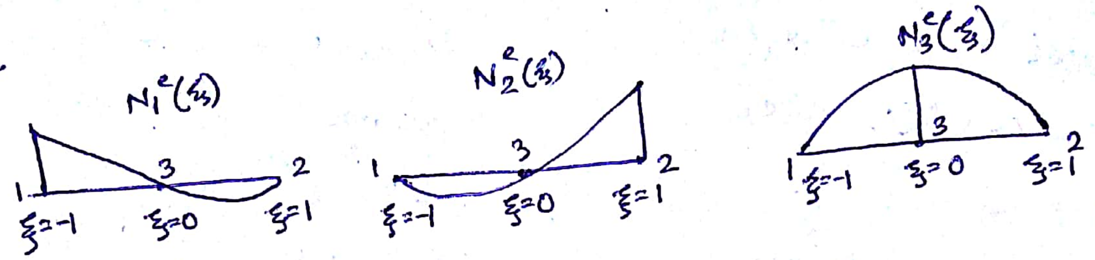
CSMD

Assignment-05

– Anurag Bhattacharjee

5.1




(a)

Given,

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2$$

From the shape-function,

for node '3' ($\xi=0$), $N_1^e=0$

$$\therefore N_1^e(\xi) = a_0 + a_1(0) + a_2(0)^2$$

$$\Rightarrow 0 = a_0$$

$$\therefore a_0 = 0$$

At node '1', $\xi=-1$ & $N_1^e=1$

$$\therefore N_1^e = a_1(-1) + a_2(-1)^2$$

$$\Rightarrow 1 = a_2 - a_1 \quad \text{— (1)}$$

&, At node '2', $\xi=1$ & $N_1^e=0$

$$\therefore N_1^e = a_1(1) + a_2(1)^2$$

$$\Rightarrow 0 = a_1 + a_2 \quad \text{— (2)}$$

Adding (1) & (2), we get

$$2a_2 = 1 \Rightarrow a_2 = 1/2$$

$$\therefore a_1 = -a_2 = -1/2$$

$$\therefore a_0 = 0,\ a_1 = -1/2,\ a_2 = 1/2$$

$$\therefore \boxed{N_1^e(\xi) = -\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^2}$$

Given,

$$N_2^e(\xi) = b_0 + b_1\xi + b_2(\xi)^2$$

From the shape function,

At node '3', $\xi=0$, $N_2^e=0$

$$\therefore N_2^e = b_0 + b_1(0) + b_2(0)^2$$

$$\Rightarrow 0 = b_0$$

At node '1', $\xi=-1$, & $N_2^e=0$

$$\therefore N_2^e = b_1(-1) + b_2(-1)^2$$

$$\Rightarrow 0 = b_2 - b_1 \quad \text{— (3)}$$

At node '2', $\xi=1$, $N_2^e=1$

$$\therefore N_2^e = b_2(1) + b_2(1)^2$$

$$\Rightarrow 1 = b_1 + b_2 \quad \text{— (4)}$$

Adding (3) & (4), we get

$$2b_2 = 1$$

$$\Rightarrow b_2 = 1/2$$

$$\therefore b_1 = b_2 = 1/2$$

$$\therefore b_0 = 0,\ b_1 = 1/2,\ b_2 = 1/2$$

$$\therefore N_2^e(\xi) = \tfrac{1}{2}\xi + \tfrac{1}{2}\xi^2$$

Also given,

$$N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

At node '3', $\xi = 0$, $N_3^e = 1$

$\therefore N_3^e = c_0 + c_1(0) + c_2(0)^2$

$\Rightarrow 1 = c_0$

At node '1', $\xi = -1$, $N_3^e = 0$

$\therefore N_3^e = 1 + c_1(-1) + c_2(-1)^2$

$\Rightarrow 0 = 1 - c_1 + c_2$

$\Rightarrow c_1 - c_2 = 1$ — (5)

At node '2', $\xi = 1$, $N_3^e = 0$

$\therefore N_3^e = 1 + c_1(1) + c_2(1)^2$

$\Rightarrow c_1 + c_2 = -1$ — (6)

Adding (5) & (6), we get

$2c_1 = 0 \Rightarrow c_1 = 0$

& $c_2 = 0 - 1 = -1$

$\therefore N_3^e(\xi) = 1 + (-1)\xi^2$

$\Rightarrow \boxed{N_3^e(\xi) = 1 - \xi^2}$

(b) Adding $N_1^e$, $N_2^e$ & $N_3^e$ we have-

$N_1^e + N_2^e + N_3^e$

$= \cancel{-\frac{1}{2}\xi} + \frac{1}{2}\xi^2 + \cancel{\frac{1}{2}\xi} + \frac{1}{2}\xi^2 + 1 - \xi^2$

$= \cancel{\xi^2} + 1 - \cancel{\xi^2}$

$= 1$

$\therefore$ Their sum is identically one.

(c) Derivatives of the shape functions w.r.t natural coordinates -

$N_1^e(\xi) = -\frac{1}{2}\xi + \frac{1}{2}\xi^2$

$$\Rightarrow \boxed{\frac{dN_1^e}{d\xi} = -\frac{1}{2} + \xi}$$

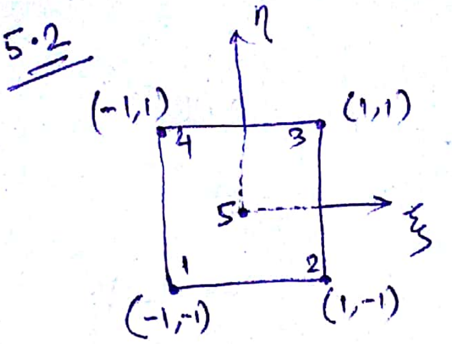
And, $N_2^e(\xi) = \frac{1}{2}\xi + \frac{1}{2}\xi^2$

$$\Rightarrow \boxed{\frac{dN_2^e}{d\xi} = \frac{1}{2} + \xi}$$

Also, $N_3^e(\xi) = 1 - \xi^2$

$$\Rightarrow \boxed{\frac{dN_3^e}{d\xi} = -2\xi}$$

5.2




Using line product method we can find shape function at node 5 as follows:-

$$N_5^e = c_5 (\eta+1)(\eta-1)(\xi+1)(\xi-1)$$

$$\Rightarrow N_5^e = c_5 (\eta^2-1)(\xi^2-1)$$

$N_5^e$ should be 1 at node 5 (0,0)

$$\therefore 1 = c_5 (0-1)(0-1)$$

$$\Rightarrow c_5 = 1 \quad \therefore \boxed{N_5^e = (\eta^2-1)(\xi^2-1)}$$

Using the same method the shape functions for the other nodes have been determined as follows:-

For the 4-node quadrilateral

$$\underline{N}_1^e = \frac{1}{4}(1-\xi)(1-\eta)$$

$$\underline{N}_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$

$$\underline{N}_3^e = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\underline{N}_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

Now, According to the question we define the shape functions at the nodes as— (For the 5-noded quadrilateral)

$$N_i^e = \underline{N}_i^e + \alpha N_5^e$$

$$\Rightarrow N_i^e = \underline{N}_i^e + \alpha (\eta^2-1)(\xi^2-1)$$

Now, In order to vanish $N_i$ at node 5, we get the following values of $\alpha$ -

<u>Node - 1</u>

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta) + \alpha(\eta^2-1)(\xi^2-1)$$

$$N_1(0,0) = 0 = \tfrac{1}{4} + \alpha \Rightarrow \boxed{\alpha = -\tfrac{1}{4}}$$

<u>Node - 2</u>

$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta) + \alpha(\eta^2-1)(\xi^2-1)$$

$$\Rightarrow N_2(0,0) = \tfrac{1}{4} + \alpha = 0$$

$$\Rightarrow \boxed{\alpha = -\tfrac{1}{4}}$$

<u>Node - 3</u>

$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta) + \alpha(\eta^2-1)(\xi^2-1)$$

$$\Rightarrow N_3(0,0) = \tfrac{1}{4} + \alpha = 0 \Rightarrow \boxed{\alpha = -\tfrac{1}{4}}$$

<u>Node - 4</u>

$$N_4 = \tfrac{1}{4}(1-\xi)(1+\eta) + \alpha(\eta^2-1)(\xi^2-1)$$

$$\Rightarrow N_4(0,0) = \tfrac{1}{4} + \alpha = 0 \Rightarrow \boxed{\alpha = -\tfrac{1}{4}}$$

$\therefore$ The values of shape functions for a five-noded quadrilateral are as follows :-

$$N_1 = \tfrac{1}{4}\left[(1-\xi)(1-\eta) - \cancel{\tfrac{1}{4}}(\eta^2-1)(\xi^2-1)\right]$$

$$N_2 = \tfrac{1}{4}\left[(1+\xi)(1-\eta) - (\eta^2-1)(\xi^2-1)\right]$$

$$N_3 = \tfrac{1}{4}\left[(1+\xi)(1+\eta) - (\eta^2-1)(\xi^2-1)\right]$$

$$N_4 = \tfrac{1}{4}\left[(1-\xi)(1+\eta) - (\eta^2-1)(\xi^2-1)\right]$$

$$N_5 = (\eta^2-1)(\xi^2-1)$$

Adding all the shape functions we have -

$$N_1^e + N_2^e + N_3^e + N_4^e + N_5^e$$

$$= \frac{1}{4}(1 - \cancel{\xi} - \cancel{\eta} + \cancel{\xi\eta}) - \frac{1}{4}\cancel{(\eta^2-1)(\xi^2-1)} + \frac{1}{4}(1 + \cancel{\xi} - \cancel{\eta} - \cancel{\xi\eta}) +$$

$$- \frac{1}{4}\cancel{(\eta^2-1)(\xi^2-1)} + \frac{1}{4}(1 + \cancel{\xi} + \cancel{\eta} + \cancel{\xi\eta}) - \frac{1}{4}\cancel{(\eta^2-1)(\xi^2-1)}$$

$$+ \frac{1}{4}(1 - \cancel{\xi} + \cancel{\eta} - \cancel{\eta\xi}) - \frac{1}{4}\cancel{(\eta^2-1)(\xi^2-1)} + \cancel{(\eta^2-1)(\xi^2-1)}$$

$$= 4 \times 1/4 = 1$$

$$\therefore \boxed{N_1^e + N_2^e + N_3^e + N_4^e + N_5^e = 1}$$

5.3

Let, $n \rightarrow$ number of nodes

$n_F \rightarrow$ number of element degrees of freedom

$n_R \rightarrow$ number of independent rigid body modes

$n_g \rightarrow$ number of gauss points

$n_E \rightarrow$ order of stress-strain matrix.

To attain rank-sufficiency,

$$\boxed{n_E n_g \geq n_F - n_R}$$

For a hexahedron, $n_E = 6$, $n_R = 6$ $\therefore \boxed{n_g \geq \frac{n_F - 6}{6}}$

$n_F = n \times 3$

| Element | $n$ | $n_F (n \times 3)$ | $n_F - 6$ | Min $n_g \left(\frac{n_F - 6}{6}\right)$ | Recommended Rule |
|---|---|---|---|---|---|
| 8-node hexahedron | 8 | 24 | 18 | 3 | 2×2×2 |
| 20-node hexahedron | 20 | 60 | 54 | 9 | 3×3×3 |
| 27 node hexahedron | 27 | 81 | 75 | 12.5 | 3×3×3 |
| 64 node hexahedron | 64 | 192 | 186 | 31 | 4×4×4 |