Computational Structural Mechanics and Dynamics

Assignment 5

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Assignment 5.1

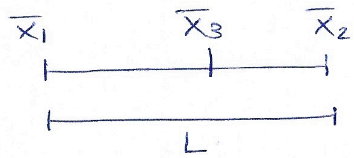
The isoperimetric definition of the straight-node bar element in its local system x is

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

Here ξ is the isoparametric coordinate that takes the values -1, 1 and 0 at the nodes 1, 2 and 3 respectively, while N_1^e, N_2^e and N_3^e are the shape functions for a bar element. For simplicity, take $\overline{x_1} = 0,\ \overline{x_2} = L$ and $\overline{x_3} = {}^1/_2\, L + \alpha L$. Here L is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = 1/2L$.

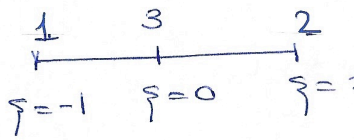
Show that the minimum α (minimal in absolute value sense) for which $J = {}^{d\bar{x}}/_{d\xi}$ vanishes at a point in the element are $\pm 1/4$ (the quarter points). Interpret this result as a singularity by showing that the axial strain becomes infinity at an end point.

* CARTESIAN COORDINATES:



$$\begin{cases} \bar{x}_1 = 0 \\ \bar{x}_2 = L \\ \bar{x}_3 = \frac{1}{2}L + \alpha L \end{cases}$$

* NATURAL COORDINATES:
(1D QUADRATIC ELEMENT)



- The Jacobian, can be defined as $J = \frac{d\bar{x}}{d\xi}$, so that, the $\bar{x}$ is going to be determined, and the Jacobian is going to be calculated in order to obtain the α values for which $J = 0$

1. The $\bar{x}$ is defined as: $\bar{x} = \bar{x}_1 N_1^e + \bar{x}_2 N_2^e + \bar{x}_3 N_3^e$
2. Working with a 1D quadratic element, the shape functions $N = a + b\xi + c\xi^2$ being ξ, the natural coordinate of the point. Substituting the ξ values at each point the constants values (a, b, c) are determined in each case, being the shape functions:

$$\begin{cases} N_1^e = \frac{1}{2}\xi(\xi - 1) \\ N_2^e = \frac{1}{2}\xi(1 + \xi) \\ N_3^e = 1 - \xi^2 \end{cases}$$

3. Introducing in the $\bar{x}$ definition the 3 shape functions and the cartesian coordinates values $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$:

$$\bar{x} = \left[\frac{1}{2}\xi(\xi-1)\right]0 + \left[\frac{1}{2}\xi(1+\xi)\right]L + \left[1-\xi^2\right]\left(\frac{1}{2}L + \alpha L\right) =$$

$$= \frac{1}{2}\xi(1+\xi)L + \frac{1}{2}L(1-\xi^2) + \alpha L(1-\xi^2)$$

4. Now, the Jacobian can be calculated

$$J = \frac{d\bar{x}}{d\xi} = \left(\frac{1}{2} + \cancel{\frac{1}{2}2\xi}\right)L + \left(\cancel{\frac{-2\xi}{2}}L\right) + \left(-2\xi\alpha L\right)$$

$$= \frac{L}{2} - 2\xi\alpha L$$

5. Applying the condition $J = 0$

$$J = 0 \rightarrow 0 = \frac{L}{2} - 2\xi\alpha L$$

$$-2\xi\alpha = -\frac{1}{2} \rightarrow \xi\alpha = \frac{1}{4} \rightarrow \alpha = \frac{1}{4\xi}$$

- As the maximum values, taking in absolut value, that ξ can have are 1 and -1, introducing them in the previous equation, the minimum values (absolute value) that α can get are:

$$\left.\begin{array}{l} \xi = 1 \rightarrow \alpha = \frac{1}{4} \\ \xi = -1 \rightarrow \alpha = \frac{-1}{4} \end{array}\right\}$$

- So it can be conclud that the minimum α for which $J = \frac{d\bar{x}}{d\xi}$ vanishes are $\pm 1/4$

- This result can be interpreted as a singularity, by calculating the strain at the end points ($\xi = -1$ and $\xi = 1$).

1. Working with natural coordinates, the strain matrix can be writed as:

$[\varepsilon] = [B][J^{-1}][u]$, where J^{-1} is the invese of the Jacobian.

$$J = \frac{d\bar{x}}{d\xi} \rightarrow J^{-1} = \frac{d\xi}{d\bar{x}}$$

- It has been shown that, at the end points of the element, the Jacobian vanishes, so that, the axial strain ε_{xx}:

$$\text{If } J = \frac{d\bar{x}}{d\xi} = 0 \rightarrow J^{-1} = \frac{1}{J} = \frac{1}{0} = \infty$$

- As for calculate the strain, the B matrix is multiplied by the inverse of the Jacobian, (calculated as ∞), it can be concluded that the strain will be infinity too,

$$\text{As } J^{-1} = \infty, \quad \varepsilon = BJ^{-1}u \rightarrow \boxed{\varepsilon_{xx} = \infty}$$

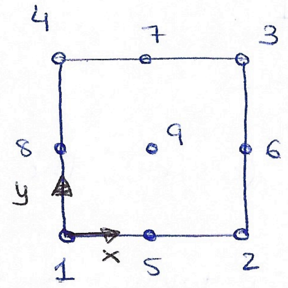
Assignment 5.2

Extend the results obtained from the previous Exercise for a 9-node plane stress element. The element is initially a perfect square, nodes 5, 6, 7 and 8 are at the midpoint of the sides 1-2, 2-3, 3-4 and 4-1 respectively and 9 at the center of the square.

Move node 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of “singular elements” for fracture mechanics.

- Now, a 9-node quadrilateral element is going to be studied, in order to check at which point the Jacobian determinant vanishes if point 5 is moved towards node 2.

- First of all, the element an its dimensions are defined



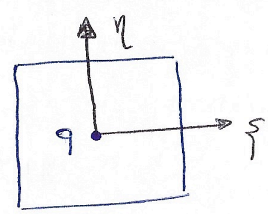
$$\begin{cases} x_1 = 0 \\ y_1 = 0 \\ L = 1 \end{cases}$$

- The length of the square is equal to one and the origin of the cartesian coordinates is fixed at point 1.

- For this kind of element, the Jacobian is defined as

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$

so that, natural coordinates are introduced, fixing the origin at point 9, where $\xi = 0$, $\eta = 0$, and knowing that any $\xi \in [-1, 1]$ and any $\eta \in [-1, 1]$



- Finally, in order to calculate the Jacobian determinant x, y and the shape functions should be defined:

  - $x = \sum_{i=1}^{9} N_i(\xi, \eta)\, x_i$

  - $y = \sum_{i=1}^{9} N_i(\xi, \eta)\, y_i$

  - The shape functions N_i, $i = 1 \dots 9$ defined for the 9-node quadrilateral element can be obtained using the studied methods.

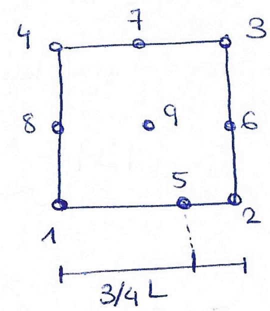
- Using the help of Matlab to solve the problem (code attached below), all the needed elements (x, y, shape functions and Jacobian) are defined in order to check at which point of line 1-2, occupaid by node 5, the $|J|$ vanishes.

- As the determinant $|J|$ is calculated at point 2, the natural coordinates ξ, η take the values: $\begin{cases} \xi = 1 \\ \eta = -1 \end{cases}$

- For node 5, the initial coordinates $(x, y) = (0'5, 0)$. The x-coordinate changes getting closer to 1.

- Runing the Matlab code, the determinant $|J|$ takes value zero when point 5 coordinates are $(x, y) = (0'75, 0)$, the quarter point of side 1-2.

At $(x_5, y_5) = (0'75, 0) \rightarrow |J| = 0$



- The obtained results, ($|J| = 0$) tell us that for 2D higher order elements the proper location of the corner nodes (1, 2, 3, 4) is not enough. The no-corner nodes, as node 5, should be placed enough close to their natural coordinates in order to avoid local distorsions. Although, this cases have applications in the constructions of special crack elements for linear fracture mechanics.

MATLAB CODE

```
syms e
syms n

%Shape functions
N1 = 0.25*(e-1)*(n-1)*e*n;
N2 = 0.25*(1+e)*(n-1)*e*n;
N3 = 0.25*(1+e)*(1+n)*e*n;
N4 = 0.25*(e-1)*(1+n)*e*n;
N5 = 0.5*(1-e^2)*(n^2-n);
N6 = 0.5*(e^2+e)*(1-n^2);
N7 = 0.5*(1-e^2)*(n^2+n);
N8 = 0.5*(e^2-e)*(1-n^2);
N9 = (1-e^2)*(1-n^2);

% x & y coordinates
% x5 = 0.75
x = 0*N1+1*N2+1*N3+0*N4+0.75*N5+1*N6+0.5*N7+0*N8+0.5*N9;
y = 0*N1+0*N2+1*N3+1*N4+0*N5+0.5*N6+1*N7+0.5*N8+0.5*N9;

%Jacobian
J = [diff(x,e) diff(y,e)
    diff(x,n) diff(y,n)];

%Determinant
subs (det(J), [e, n], [1 , -1])
```