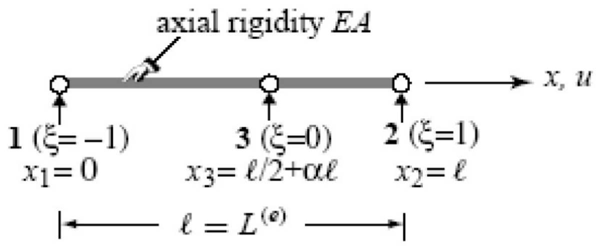
Assignment 5.1



Definition of the shape functions in ispoarametric space

$$N_1(\xi) = \frac{1}{2}\xi(\xi - 1)$$

$$N_3(\xi) = 1 - \xi^2$$

$$N_2(\xi) = \frac{1}{2}\xi(1 + \xi)$$

Expressing x as a function of ξ :

$$\bar{x} = \sum_{i=1}^{3} \bar{x}_i N_i = -\propto L\xi^2 + \frac{L}{2}\xi + \mathrm{L}(\frac{1}{2} + \alpha)$$

Therefore, the jacobian is

$$J = \frac{d\bar{x}}{d\xi} = L(\frac{1}{2} - 2\alpha\xi)$$

We have to find the minimum absolute value of α for which the Jacobian vanishes:

$$J = 0 \rightarrow L\left(\frac{1}{2} - 2\alpha\xi\right) = 0 \qquad |\alpha| = \frac{1}{4}\left|\frac{1}{\xi}\right|$$

Taking into account that $-1 \leq \xi \leq 1$, α becomes: $\alpha = \pm\frac{1}{4}$

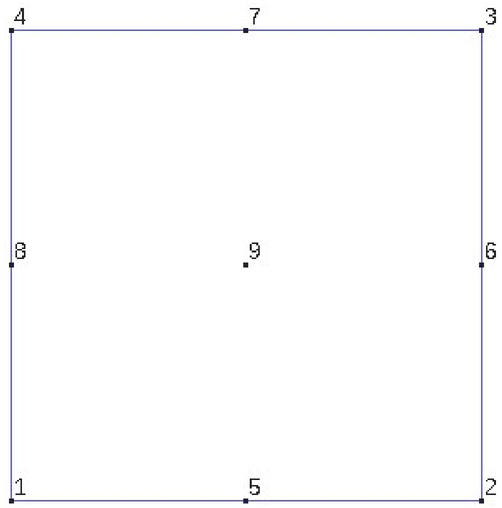
The axial strains are given by:

$$\varepsilon = \frac{d\bar{u}}{d\bar{x}} = \frac{d\bar{u}}{d\xi}\frac{d\xi}{d\bar{x}} = J^{-1}\frac{d\bar{u}}{d\xi}$$

When α = ±1/4, the Jacobian is zero and the value of the axial strains tends to infinity. Consequently, α = 1/4 is a singularity for this problem. Node 3 should remain inside the central third of the element to avoid this.

Assignment 5.2

9 node quadrilateral element, showed in the figure below:



The shape functions are the ones for a quadratic quadrilateral element in the reference element.

Node	ξi	ηi	Ni
1	-1	-1	$1/4\,(1-\xi)(1-\eta)\xi\eta$
2	1	-1	$-1/4\,(1+\xi)(1-\eta)\xi\eta$
3	1	1	$1/4\,(1+\xi)(1+\eta)\xi\eta$
4	-1	1	$-1/4\,(1-\xi)(1+\eta)\xi\eta$
5	0	-1	$-1/2\,(1-\xi^2)(1-\eta)\eta$
6	1	0	$1/2\,(1+\xi)(1-\eta^2)\xi$
7	0	1	$1/2\,(1-\xi^2)(1+\eta)\eta$
8	-1	0	$-1/2\,(1-\xi)(1-\eta^2)\xi$
9	0	0	$(1-\xi^2)(1-\eta^2)$

If we consider L to be the length of the square, the position of the nodes in global coordinates may be defined in the following table

Node	Xi	Yi
1	0	0
2	L	0
3	L	L
4	0	L
5	L/2 +αL	0
6	L	L/2
7	L/2	L
8	0	L/2
9	L/2	L/2

The jacobian in this 2D case is a 2 x2 matrix:

$$\mathbf{J} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad \frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} x_i \frac{\partial N_i^e}{\partial \xi}, \quad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} y_i \frac{\partial N_i^e}{\partial \xi}$$

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{n} x_i \frac{\partial N_i^e}{\partial \eta}, \quad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{n} y_i \frac{\partial N_i^e}{\partial \eta}$$

We have to compute the jacobian with the previous equation and substitute the particular local values of node 2, $\xi = 1, \eta = -1$.

$$J(\xi,\eta) = \frac{L}{2}\begin{bmatrix} 2\xi\eta\alpha - 2\xi\eta^2\alpha + 1 & -\alpha(2\eta - 1)(\xi^2 - 1) \\ 0 & 1 \end{bmatrix}$$

Find the value of α that minimizes the value of the determinant of the Jacobian

$$|J(\xi,\eta)| = \frac{L}{2}(2\xi\eta\alpha - 2\xi\eta^2\alpha + 1)$$

$$|J(1,-1)| = \frac{1}{4} - \alpha$$

We are facing a singularity again since it makes the Jacobian determinant go to zero. This means that for the quadratic quadrilateral case, the nodes located on the edges of the element should remain inside the central third of the element.