Assignment 5.

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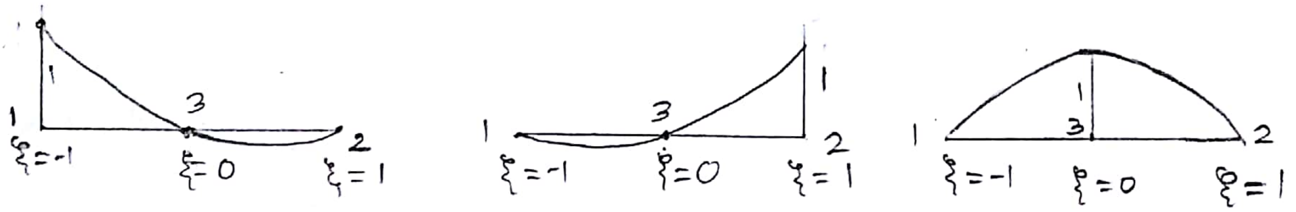
Problem 5.1

Given Three node bar element with three shape functions.

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2$$

$$N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2$$

$$N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$



(a) $\Rightarrow$ Node 1:

We know for $\xi = -1$, $N_1^e(\xi) = 1$ and 0 for other values of ξ.

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2$$

when $\xi = -1$, $\quad 1 = a_0 - a_1 + a_2 \quad \cdots (1)$

when $\xi = 0$, $\quad 0 = a_0 - 0 + 0 \quad \Rightarrow a_0 = 0 \quad \cdots (2)$

when $\xi = 1$, $\quad 0 = a_0 + a_1 + a_2 \quad \cdots$

By solving above three equations we get

$$\Rightarrow a_0 = 0, \quad a_1 = -1/2, \quad a_2 = \frac{1}{2}$$

Node 2:

Again, we know for $\xi = 1$, $N_2^e(\xi) = 1$ and 0 for other values of ξ.

$$N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2$$

when $\xi = -1$, $\quad 0 = b_0 - b_1 + b_2 \quad \cdots (4)$

when $\xi = 0$, $\quad 0 = b_0 + 0 + 0. \Rightarrow b_0 = 0 \quad \cdots (5)$

when $\xi = 1$, $\quad 1 = b_0 + b_1 + b_2 \quad \cdots (6)$

from solving above three equations. we get

$\Rightarrow b_0 = 0, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{2}$

Node 3:

we know for $\xi = 0$, $N_3^e(\xi) = 1$ and 0 for other values of ξ

$$N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

when $\xi = -1$, $\quad 0 = c_0 - c_1 + c_2 \quad \cdots (7)$

when $\xi = 0$, $\quad 1 = c_0 + 0 + 0 \quad \Rightarrow c_0 = 1 \quad \cdots (8)$

when $\xi = 1$, $\quad 0 = c_0 + c_1 + c_2 \quad \cdots (9)$

By solving above three equations, we get,

$\Rightarrow c_0 = 1, \quad c_1 = 0, \quad c_2 = -1$

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(b) $\Rightarrow$ To show,

$$N_1^e + N_2^e + N_3^e = 1$$

Adding all the shape function.

$$N_1^e(\xi) + N_2^e(\xi) + N_3^e(\xi)$$

$$= 0 + \cancel{\left(-\frac{1}{2}\right)\xi} + \cancel{\frac{1}{2}\xi^2} + 0 + \cancel{\frac{1}{2}\xi} + \cancel{\frac{1}{2}\xi^2} + 1 + 0 - \cancel{\xi^2}$$

$$= 1$$

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(c) $\Rightarrow$ Calculating the derivatives of shape functions with respect to natural co-ordinates.

$$N_1^e(\xi) = -\frac{1}{2}\xi + \frac{1}{2}\xi^2$$

$$\Rightarrow \frac{dN_1^e(\xi)}{d\xi} = -\frac{1}{2} + \xi \quad //$$

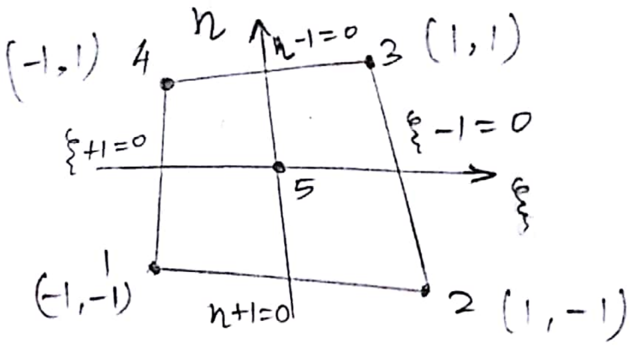
$$N_2^e(\xi) = \frac{1}{2}\xi + \frac{1}{2}\xi^2$$

$$\Rightarrow \frac{dN_2^e(\xi)}{d\xi} = \frac{1}{2} + \xi \quad //$$

$$N_3^e(\xi) = 1 - \xi^2$$

$$\Rightarrow \frac{dN_3^e(\xi)}{d\xi} = -2\xi \quad //$$

Problem 5.2



$\Rightarrow$ Using the line product method we have.

$$N_5(\xi,\eta) = c_5 L_{1-2} L_{2-3} L_{3-4} L_{4-1}$$
$$= c_5 (\eta+1)(\xi-1)(\eta-1)(\xi+1)$$

At Node 5, $N_5(\xi,\eta)=1$ and $\xi=0$, $\eta=0$

$$1 = c_5 (0+1)(0-1)(0-1)(0+1)$$
$$c_5 = 1$$

Then,

$$N_5(\xi,\eta) = (\eta+1)(\xi-1)(\eta-1)(\xi+1)$$
$$= (\xi^2-1)(\eta^2-1)$$

At Node 1

$$\underline{N_1}(\xi,\eta) = c_1 L_{2-3} L_{3-4}$$
$$= c_1 (\xi-1)(\eta-1)$$

for Node 1, $\underline{N_1}(-1,-1) = 1$

$$\Rightarrow 1 = c_1 (-1-1)(-1-1)$$
$$c_1 = 1/4$$

For Node 2

$$\underline{N}_2(\xi,\eta) = c_2\, L_{3-4}\, L_{4-1}$$
$$= c_2\,(\eta-1)(\xi+1)$$

At Node 2, $\underline{N}_2(1,-1) = 1$

$$\Rightarrow 1 = c_2\,(-1-1)(1+1)$$
$$c_2 = -1/4$$

$$\Rightarrow \underline{N}_2(\xi,\eta) = -\frac{1}{4}(\eta-1)(\xi+1)$$

For Node 3

$$\underline{N}_3(\xi,\eta) = c_3\, L_{4-1}\, L_{1-2}$$
$$= c_3\,(\xi+1)(\eta+1)$$

At Node 3, $\underline{N}_3(1,1) = 1$

$$\Rightarrow 1 = c_3(1+1)(1+1)$$
$$c_3 = \frac{1}{4}$$

$$\Rightarrow \underline{N}_3(\xi,\eta) = \frac{1}{4}(\xi+1)(\eta+1)$$

For Node 4

$$\underline{N}_4(\xi,\eta) = C_4 (L_{1-2} \; L_{2-3})$$
$$= C_4 (\eta+1)(\xi-1)$$

At Node 4, $\underline{N}_4(-1,1) = 1$

$$\Rightarrow 1 = C_4 (1+1)(-1-1)$$
$$C_4 = -\frac{1}{4}$$

$$\Rightarrow \underline{N}_4(\xi,\eta) = -\frac{1}{4}(\eta+1)(\xi-1)$$

The corner shape function is given by

$$N_i = \underline{N}_i + \alpha N_5$$
$$N_2 = -\frac{1}{4}(\eta-1)(\xi+1) + \alpha(\xi^2-1)(\eta^2-1)$$

At Node 5, $N_2 = 0$, $\eta = 0$, $\xi = 0$.

$$\Rightarrow 0 = -\frac{1}{4}(0-1)(0+1) + \alpha(0-1)(0-1)$$

$$\Rightarrow 0 = \frac{1}{4} + \alpha$$
$$\Rightarrow \alpha = -\frac{1}{4} \;//$$

The shape functions are presented below

$$N_1 = \underline{N_1} + \alpha N_5$$
$$= \frac{1}{4}(\xi - 1)(\eta - 1) - \frac{1}{4}(\xi^2 - 1)(\eta^2 - 1)$$

$$N_2 = \underline{N_2} + \alpha N_5$$
$$= -\frac{1}{4}(\eta - 1)(\xi + 1) + \left(-\frac{1}{4}\right)(\xi^2 - 1)(\eta^2 - 1)$$

$$N_3 = \underline{N_3} + \alpha N_5$$
$$= \frac{1}{4}(\xi + 1)(\eta + 1) - \frac{1}{4}(\xi^2 - 1)(\eta^2 - 1)$$

$$N_4 = \underline{N_4} + \alpha N_5$$
$$= -\frac{1}{4}(\eta + 1)(\xi - 1) + \left(-\frac{1}{4}\right)(\xi^2 - 1)(\eta^2 - 1)$$

Now, To show,

$$N_1 + N_2 + N_3 + N_4 + N_5 = 1$$

$$\Rightarrow \quad N_1 + N_2 + N_3 + N_4 + N_5$$
$$= \frac{1}{4}(\xi - 1)(\eta - 1) - \cancel{\frac{1}{4}(\xi^2 - 1)(\eta^2 - 1)} - \frac{1}{4}(\eta - 1)(\xi + 1)$$
$$- \cancel{\frac{1}{4}(\xi^2 - 1)(\eta^2 - 1)} + \frac{1}{4}(\xi + 1)(\eta + 1) - \cancel{\frac{1}{4}(\xi^2 - 1)(\eta^2 - 1)}$$
$$- \frac{1}{4}(\eta + 1)(\xi - 1) - \cancel{\frac{1}{4}(\xi^2 - 1)(\eta^2 - 1)} + \cancel{(\xi^2 - 1)(\eta^2 - 1)}$$

$$= \frac{1}{4}\Big[(\xi - 1)(\eta - 1) - (\eta - 1)(\xi + 1) + (\xi + 1)(\eta + 1)$$
$$- (\eta + 1)(\xi - 1)\Big]$$

$$= \frac{1}{4}\left[(n-1)\left(\cancel{n}-1-\cancel{n}-1\right)+(n+1)\left(\cancel{n}+1-\cancel{n}+1\right)\right]$$

$$= \frac{1}{4}\left[(n-1)(-2)+(n+1)\,2\right]$$

$$= \frac{1}{4}\left[-\cancel{2n}+2+\cancel{2n}+2\right]$$

$$= \frac{1}{4}[4]$$

$$= 1$$

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Verified //

Problem 5.3

1. the 8-node hexahedron

⇒ The Required gauss rule to obtain rank sufficiency is $2 \times 2 \times 2$ rule, since $(2^3 \times 6 = 48) > (8 \times 3 - 6 = 18)$ gives a rank sufficient stiffness matrix.

2. The 20-node hexahedron.

⇒ The Required gauss rule to obtain rank sufficiency is $3 \times 3 \times 3$ rule, since $(3^3 \times 6 = 162) > (20 \times 3 - 6 = 54)$ gives a rank sufficient stiffness matrix

3. The 27-node hexahedron

⇒ The Required gauss rule to obtain rank sufficiency is $3 \times 3 \times 3$ rule since, $(3^3 \times 6 = 162) > (27 \times 3 - 6 = 75)$ gives a rank sufficient stiffness matrix

(4) The 64-node hexahedron

⇒ The required gauss rule to obtain rank Sufficiency is $4 \times 4 \times 4$ rule, since

$$(4^3 \times 6 = 384) > (64 \times 3 - 6 = 186)$$ gives a rank sufficient stiffness matrix