Universitat Politècnica de Catalunya
Master of Science in Computational Mechanics
Computational Strutural Mechanics and Dynamics
CSMD
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Assignment 5 - Isoparametric Representation and Convergence Requirements

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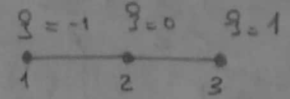
5.1

a) Given:



$N_1(\xi) = a_0 + a_1\xi + a_2\xi^2$; $N_2(\xi) = b_0 + b_1\xi + b_2\xi^2$

$N_3(\xi) = c_0 + c_1\xi + c_2\xi^2$

for N_1, the compatibility vezos:

$N_1(-1) = 1 = a_0 - a_1 + a_2$

$N_1(0) = 0 = a_0$

$N_1(1) = 0 = a_0 + a_1 + a_2$

$a_0 = 0$; $1 = 2a_2 \Rightarrow a_2 = \frac{1}{2} \Rightarrow a_1 = \frac{-1}{2}$

thus:

$$\boxed{N_1(\xi) = \frac{1}{2}(\xi - 1)\xi}$$

for N_2, the compatibility vezos:

$N_2(-1) = 0 = b_0 - b_1 + b_2$

$N_2(0) = 1 = b_0$

$N_2(1) = 0 = b_0 + b_1 + b_2$

$b_0 = 1$; $-2 = 2b_2 \Rightarrow b_2 = -1$; $b_1 = 0$

thus:

$$\boxed{N_2(\xi) = 1 - \xi^2}$$

for N_3, the compatibility vezos:

$N_3(-1) = 0 = c_0 - c_1 + c_2$; $c_0 = 0$; $2c_2 = 1 \Rightarrow c_2 = \frac{1}{2} \Rightarrow c_1 = \frac{1}{2}$

$N_3(0) = 0 = c_0$

$N_3(1) = 1 = c_0 + c_1 + c_2$

thus:

$$\boxed{N_3(\xi) = \frac{1}{2}(\xi + 1)\xi}$$

b)

$$N_1(\xi)+N_2(\xi)+N_3(\xi)=\frac{1}{2}(\xi-1)\xi+(1-\xi^2)+\frac{1}{2}(\xi+1)\xi$$

$$=\frac{1}{2}\xi^2-\cancel{\frac{\xi}{2}}+1-\xi^2+\cancel{\frac{\xi}{2}}+\frac{1}{2}\xi^2$$

$$=\cancel{\frac{1}{2}\xi^2+\frac{1}{2}\xi^2-\xi^2}+1$$

$$N_1(\xi)+N_2(\xi)+N_3(\xi)=1$$

c)

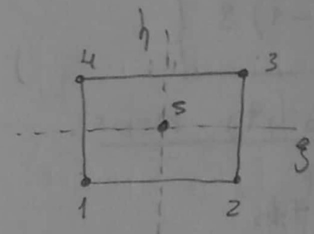
$$\frac{dN_1}{d\xi}=\frac{d}{d\xi}\left(\frac{1}{2}\xi^2-\frac{1}{2}\xi\right)=\xi-\frac{1}{2}$$

$$\frac{dN_2}{d\xi}=\frac{d}{d\xi}\left(1-\xi^2\right)=-2\xi$$

$$\frac{dN_3}{d\xi}=\frac{d}{d\xi}\left(\frac{\xi^2}{2}+\frac{\xi}{2}\right)=\xi+\frac{1}{2}$$

5.2

by the Line method we Propose:



$$N_5(\xi,\eta)=c_1L_1L_2\dots L_m$$

$$N_5(\xi,\eta)=c_1L_{1-2}L_{2-3}L_{3-4}L_{4-1}$$

where:

along $L_{1-2}:\ \eta=-1 \Rightarrow L_{1-2}=\eta+1$

along $L_{2-3}:\ \xi=1 \Rightarrow L_{1-3}=\xi-1$

along $L_{4-3}:\ \eta=1 \Rightarrow L_{4-3}=\eta-1$

along $L_{4-1}:\ \xi=-1 \Rightarrow L_{4-1}=\xi+1$

thus:

$$N_5=c_1(1+\xi)(1-\xi)(1+\eta)(1-\eta)$$

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for compatibility

$N_5(0,0) = 1 = C_1 \quad \Rightarrow \quad C_1 = 1$

$N_5 = (1+\xi)(1-\xi)(1+\eta)(1-\eta)$

$\boxed{N_5 = (1-\xi^2)(1-\eta^2)}$

for the 4 corner nodes we have:

$\bar{N}_1 = C_1 L_{2-3} L_{4-3} = C_1(\xi-1)(\eta-1)$

$\bar{N}_1(-1,-1) = 1 = 4C_1 \quad \Rightarrow \quad C_1 = \frac{1}{4}$

$\boxed{\bar{N}_1 = \frac{1}{4}(1-\xi)(1-\eta)}$

$\bar{N}_2 = C_2 L_{3-4} L_{4-1} = C_2(\xi+1)(\eta-1)$

$\bar{N}_2(1,-1) = 1 = -4C_2 \quad \Rightarrow \quad C_2 = -\frac{1}{4}$

$\boxed{\bar{N}_2 = \frac{1}{4}(1+\xi)(1-\eta)}$

$\bar{N}_3 = C_3 L_{4-1} L_{1-2} = C_3(1+\xi)(1+\eta)$

$\bar{N}_3(1,1) = 1 = 4C_3 \quad \Rightarrow \quad C_3 = \frac{1}{4}$

$\boxed{\bar{N}_3 = \frac{1}{4}(1+\xi)(1+\eta)}$

$\bar{N}_4 = C_4 L_{1-2} L_{2-3} = C_4(\xi-1)(1+\eta)$

$\bar{N}_4(-1,1) = 1 = -4C_4 \Rightarrow \quad C_4 = -\frac{1}{4}$

$\boxed{\bar{N}_4 = \frac{1}{4}(1-\xi)(1+\eta)}$

for full compatibility N_1, N_2, N_3 and N_4 must vanish at node 5:
using the hierarchical correction technique we can write:

$N_i = \bar{N}_i + \alpha N_5 = \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2)$

$N_1(0,0) = 0 = \frac{1}{4} + \alpha \quad \Rightarrow \quad \alpha = -\frac{1}{4}$

same α value is found for N_2, N_3 and N_4

thus:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$

$$= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi)(1-\eta)(1+\xi)(1+\eta)$$

$$\boxed{N_1 = \frac{1}{4}(1-\xi)(1-\eta)\left[1-(1+\xi)(1+\eta)\right]}$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1+\xi)(1-\eta)(1-\xi)(1+\eta)$$

$$\boxed{N_2 = \frac{1}{4}(1+\xi)(1-\eta)\left[1-(1-\xi)(1+\eta)\right]}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1+\xi)(1+\xi)(1-\xi)(1-\eta)$$

$$\boxed{N_3 = \frac{1}{4}(1+\xi)(1+\eta)\left[1-(1-\xi)(1-\eta)\right]}$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi)(1+\eta)(1+\xi)(1-\eta)$$

$$\boxed{N_4 = \frac{1}{4}(1-\xi)(1+\eta)\left[1-(1+\xi)(1-\eta)\right]}$$

$$\boxed{N_5 = (1-\xi^2)(1-\eta^2)}$$

- Interpolation condition is now satisfied by all $N_i(\xi,\eta)$
- Local suport condition is also fullfilled by N_1, N_2, N_3 and N_4 as they vanish in boundaries where i node doesn't appear
- Interelement compatibility condition is proved also, as follows:

for N_1: at L_{1-2} $\eta = cte = -1 \Rightarrow N_1 = \frac{1}{2}(1-\xi)$ (Linear)

at L_{4-1} $\xi = cte = -1 \Rightarrow N_1 = \frac{1}{2}(1-\eta)$ (Linear) for N_2: at L_{1-2} $\eta = -1$ $\Rightarrow N_2 = \frac{1}{2}(1+\xi)$ (Linear)

at L_{2-3} $\xi = 1$ $\Rightarrow N_2 = \frac{1}{2}(1-\eta)$ (Linear)

for N_3: at L_{2-3} $\xi = 1$ $\Rightarrow N_3 = \frac{1}{2}(1+\eta)$ (Linear)

at L_{4-3} $\eta = 1$ $\Rightarrow N_3 = \frac{1}{2}(1+\xi)$ (Linear)

for N_4: at L_{4-3} $\eta = 1$ $\Rightarrow N_4 = \frac{1}{2}(1-\xi)$ (Linear)

at L_{4-1} $\xi = -1$ $\Rightarrow N_4 = \frac{1}{2}(1+\eta)$ (Linear)

- thus, at boundaries all $N_i(\xi,\eta)$ are linear functions (1^{st} order) which requires 2 nodes:

- the summation of the shape functions is given by:

$$N_1+N_2+N_3+N_4+N_5 = \overbrace{\frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}^{N_1} +$$
$$+ \underbrace{\frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}_{N_2} + \underbrace{\frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}_{N_3} +$$
$$+ \underbrace{\frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)}_{N_4} + (1-\xi^2)(1-\eta^2) \quad N_5$$

$$N_1+N_2+N_3+N_4+N_5 = \frac{1}{4}\left[(1-\xi)(1-\eta)+(1+\xi)(1-\eta)+(1+\xi)(1+\eta)+(1-\xi)(1+\eta)\right] +$$
$$+ (1-\xi^2)(1-\eta^2)\underbrace{\cancel{\left[1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}\right]}}_{0}$$
$$= \frac{1}{4}\left\{(1-\xi)\left[(1-\cancel{\eta})+(1+\cancel{\eta})\right]+(1+\xi)\left[(1-\cancel{\eta})(1+\cancel{\eta})\right]\right\}$$
$$= \frac{1}{4}\left[2(1\cancel{-\xi})+2(1\cancel{+\xi})\right] = \frac{1}{4}(2+2) = 1$$

thus:

$$\boxed{\sum_{i=1}^{5} N_i = 1}$$

5.3

Given that, for a rank sufficiency elemental stiffness matrix we have:

$$r = m_f - m_R$$

where m_f is the number of degrees of freedom of the element, and m_R the number of independent rigid body modes.

as each gauss point adds m_E to the rank of $\underline{\underline{K}}^e$, to attain rank sufficiency, a numerical integration by gauss quadrature must use a number of gauss points m_G such that

$$m_E m_G \geq m_f - m_R$$

here we define m_E as the rank of the elasticity matrix denoted as $\underline{\underline{E}}$.

1) <u>8-node hexahedron</u>

- 3D element
- $m_E = 6$ ($\underline{\underline{E}}$ is a 6x6 matrix)
- $m_f = 3 \times 8 = 24$ (3 degree of freedom per node)
- $m_R = 6$ (3 rotations and 3 translations)

thus:

$$r = m_f - m_R = 24 - 6 = 18$$

$$6 m_G \geq 18 \quad \Rightarrow \quad m_G \geq 3$$

hence: a gauss rule <u>1x1x1</u> provides rank sufficient matrix

2) <u>20-node hexahedron</u>

$m_E = 6$

$m_R = 6$

$m_f = 3 \times 20 = 60$

thus:

$$r = m_f - m_R = 60 - 6 = 54$$

$$6 m_G \geq 54$$

$$m_G \geq 9$$

hence: a gauss rule 2x2x2 gives only 8 points, then, a gauss rule <u>3x3x3</u> is necessary

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3) <u>27-node hexahedron</u>

$m_E = 6$

$m_R = 6$

$m_f = 3 \times 27 = 81$

thus:

$r = m_f - m_R = 81 - 6 = 75$

$6 m_G \geqslant 75$

$m_G \geqslant 12{,}5$

hence, again a Gauss rule <u>3x3x3</u> is needed

4) <u>64-node hexahedron</u>

$m_E = 6$

$m_R = 6$

$m_f = 3 \times 64 = 192$

thus:

$r = m_f - m_R = 192 - 6 = 186$

$6 m_G \geqslant 186$

$m_G \geqslant 31$

hence, a Gauss quadrature rule 3x3x3 only provides $m_G = 27$, thus a Gauss rule <u>4x4x4</u> is needed