Computational Structural Mechanics and Dynamics

Assignment 5

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Problem 5.1

Consider a three-node bar element referred to the natural coordinate ξ. The two end nodes and the mid node are identified as 1, 2 and 3 respectively. The natural coordinates of nodes 1, 2 and 3 are $\xi = -1$, $\xi = 1$ and $\xi = 0$, respectively. The variation of the shape functions $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ is sketched in the figure below. These functions must be quadratic polynomials in ξ:

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2 \qquad N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2 \qquad N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$$

(a) Determine the coefficients $a0, ..., c2$ using the node value conditions depicted in figure. For example $N_1^e = 1$ for $\xi = 1$ and 0 for the rest of natural coordinates. The rest of the nodes follow the same scheme.
<u>Solution</u>

$$\begin{array}{ccc} N_1^e(-1) = a_0 - a_1 + a_2 = 1 & N_2^e(-1) = b_0 - b_1 + b_2 = 0 & N_3^e(-1) = c_0 - c_1 + c_2 = 0 \\ N_1^e(0) = a_0 = 0 & N_2^e(0) = b_0 = 0 & N_3^e(0) = c_0 = 1 \\ N_1^e(1) = a_0 + a_1 + a_2 = 0 & N_2^e(1) = b_0 + b_1 + b_2 = 1 & N_3^e(1) = c_0 + c_1 + c_2 = 0 \end{array}$$

Solving the linear system we can know the values for $a0, ..., c2$:

$$N_1^e(\xi) = \frac{1}{2}\xi(1-\xi) \qquad N_2^e(\xi) = \frac{1}{2}\xi(1+\xi) \qquad N_3^e(\xi) = 1 - \xi^2$$

(b) Verify that their sum is identically one.
<u>Solution</u>

$$N_1 + N_2 + N_3 = \frac{1}{2}\xi - \frac{1}{2}\xi + \frac{1}{2}\xi^2 + \frac{1}{2}\xi^2 - \xi^2 + 1 = 1$$

(c) Calculate their derivatives respect to the natural coordinates.
<u>Solution</u>

$$\frac{\partial N_1}{\partial \xi} = \xi - \frac{1}{2} \qquad \frac{\partial N_2}{\partial \xi} = \xi + \frac{1}{2} \qquad \frac{\partial N_3}{\partial \xi} = 2\xi$$

Problem 5.2

A five node quadrilateral element has the nodal configuration shown if the figure with two perspective views of N_1^e and N_5^e. Find five shape functions $N_i^e, i = 1, ..., 5$ that satisfy compatibility and also verify that their sum is unity.

<u>Solution</u>

According to the figure, the shape function N_5 is obtained as the following:

$$N_5 = C_5 L_{12} L_{23} L_{34} L_{41} \Longrightarrow C_5(1-\xi)(1-\eta)(1+\xi)(1+\eta)$$

In order to find C_5 we should substitute the coordinates of node 5, {0,0}, in the equation. So, C_5 is found to be equal to 1.

$$N_5 = (1-\xi^2)(1-\eta^2)$$

The shape-functions for corner nodes are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta) \qquad N_3 = \frac{1}{4}(1+\xi)(1+\eta) \qquad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

which can be written in a general form as:

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

Therefore, the following formula holds for all nodes:

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) + \alpha N_5$$

in which α can be found substituting the coordinates of node 5 into any of the shape functions and setting it equal to zero. For instance, substitution of $\xi = \eta = 0$ in N_1 yields:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) + \alpha(1-\xi^2)(1-\eta^2) \Longrightarrow N_1 = \frac{1}{4} + \alpha = 0 \Longrightarrow \alpha = -\frac{1}{4}$$

In addition, the summation of all shape functions is equal to 1:

$$N_1+N_2+N_3+N_4+N_5 = \frac{1}{4}(1-\xi)(1-\eta)+\frac{1}{4}(1+\xi)(1-\eta)+\frac{1}{4}(1+\xi)(1+\eta)+\frac{1}{4}(1-\xi)(1+\eta)+4\alpha N_5+N_5$$

$$\Longrightarrow \frac{1}{4}(1-\xi-\eta+\xi\eta) + \frac{1}{4}(1+\xi-\eta-\xi\eta) + \frac{1}{4}(1+\xi+\eta+\xi\eta) + \frac{1}{4}(1-\xi+\eta-\xi\eta) - N_5 + N_5 = 1$$

Problem 5.3

Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for these elements:

1. the 8-node hexahedron
2. the 20-node hexahedron
3. the 27-node hexahedron
4. the 64-node hexahedron

Solution

The following formula is to be used for solving this problem:

$$r = min(n_F - n_R, n_E n_G)$$

where n_F, n_R, n_E and n_G are number of element DoF, number of independent rigid body modes, order of E stress-strain matrix and number of Gauss points in integration rule for K, respectively. r is the actual rank of stiffness matrix.

In our case, each of the parameters can be found as:

$$n - F = 3n; \qquad n_R = 6; \qquad n_E = 6$$

We define a parameter "rank deficiency" which has to be equal to zero, as the following:

$$d = (n_F - n_R) - r$$

According to the formulas, we start to find the number of Gauss points, n_G, for each case:

1.
$$3 \times 8 - 6 = min(3 \times 8 - 6, 6n_G) \Longrightarrow n_G = 3$$

2.
$$3 \times 20 - 6 = min(3 \times 20 - 6, 6n_G) \Longrightarrow n_G = 9$$

3.
$$3 \times 27 - 6 = min(3 \times 27 - 6, 6n_G) \Longrightarrow n_G = 13$$

4.
$$3 \times 64 - 6 = min(3 \times 64 - 6, 6n_G) \Longrightarrow n_G = 31$$