Universitat Politècnica de Catalunya

Master of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

Assignment 5
Convergence Requirements

Author:
Carlos Eduardo Ribeiro Santa Cruz Mendoza

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UNIVERSITAT POLITÈCNICA DE CATALUNYA
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1 Straight-node bar element

The quadratic bar element described on the problem is represented on Figure 1.1.

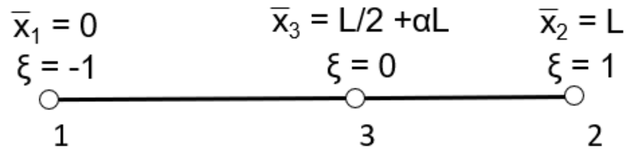

Figure 1.1: 3-noded bar element

As provided, the isoparametric relations for the 1-D element are given by Equation 1.1:

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix} \tag{1.1}$$

Whereas the quadratic shape functions are:

$$N_1 = \frac{1}{2}\xi(\xi - 1) \quad N_2 = \frac{1}{2}\xi(\xi + 1) \quad N_3 = 1 - \xi^2 \tag{1.2}$$

Inserting Equation 1.2 on Equation 1.1 and substituting the values of $\bar{x}_i$ we can find $\bar{x}$ as a function of ξ, yielding:

$$\bar{x} = \bar{x}_1 N_1 + \bar{x}_2 N_2 + \bar{x}_3 N_3 \quad \Rightarrow \quad \bar{x} = \frac{l}{2}\xi(\xi + 1) + (\frac{l}{2} + \alpha l) \cdot (1 - \xi^2) \tag{1.3}$$

The relationship found on Equation 1.3 allows the calculation of the jacobian, responsible for the mapping between $\bar{x}$ and ξ.

$$J = \frac{d\bar{x}}{d\xi} \quad \Rightarrow \quad J = l\left(\frac{1}{2} - 2\alpha\xi\right) \tag{1.4}$$

We notice that the Jacobian can reach a null value on two cases:

$$\textbf{Node 1:} \quad \xi = -1 \quad \text{and} \quad \alpha = -\frac{1}{4} \quad \Rightarrow \quad J = 0 \tag{1.5}$$

$$\textbf{Node 2:} \quad \xi = 1 \quad \text{and} \quad \alpha = \frac{1}{4} \quad \Rightarrow \quad J = 0 \tag{1.6}$$

These singularities bring consequences to the displacement field, which, from Equation 1.1, is given by:

$$u = u_1 N_1 + u_2 N_2 + u_3 N_3 \tag{1.7}$$

But from the definition of the strain we get

$$\varepsilon = \frac{du}{d\bar{x}} \quad \Rightarrow \quad \varepsilon = u_1 \frac{dN_1}{d\bar{x}} + u_2 \frac{dN_2}{d\bar{x}} + u_3 \frac{dN_3}{d\bar{x}} \tag{1.8}$$

However, from the chain rule we can state

$$\frac{dN_i}{d\bar{x}} = \frac{dN_i}{d\xi}\frac{d\xi}{d\bar{x}} = \frac{dN_i}{d\xi} J^{-1} \tag{1.9}$$

Thus, the strain is a function of the inverse of the Jacobian. This means that on the cases that the Jacobian is null, the strain would tend to infinity, representing a fracture failure.

2 Quadrilateral Element

The biquadratic element with side l described on the problem is represented on Figure 2.1.

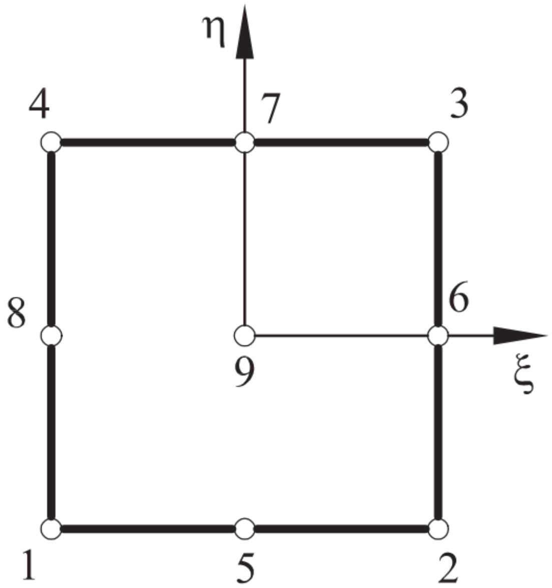

Figure 2.1: 9-node quadrilateral element

The position of the node 5 is initially $(\bar{x}, \bar{y}) = (0, -\frac{l}{2})$ and it moves tangentially to node 2, yielding coordinates $(\bar{x}, \bar{y}) = (\alpha l, -\frac{l}{2})$.

The shapes functions for the element are given by

$$\begin{aligned} N_1 &= \frac{1}{4}\xi\eta(\xi-1)(\eta-1) & N_2 &= \frac{1}{4}\xi\eta(\xi+1)(\eta-1) \\ N_3 &= \frac{1}{4}\xi\eta(\xi+1)(\eta+1) & N_4 &= \frac{1}{4}\xi\eta(\xi-1)(\eta+1) \\ N_5 &= \frac{1}{2}\eta(1-\xi^2)(\eta-1) & N_6 &= \frac{1}{2}\xi(\xi+1)(1-\eta^2) \\ N_7 &= \frac{1}{2}\eta(1-\xi^2)(\eta+1) & N_8 &= \frac{1}{2}\xi(\xi-1)(1-\eta^2) \\ & & N_9 = (1-\xi^2)(1-\eta^2) \end{aligned} \tag{2.1}$$

Similarly to the bar element, we can find the relation between the global coordinates and the isoparametric coordinates via the shape functions on Equation 2.1.

$$\bar{x} = \sum_{i=1}^{9} \bar{x}_i N_i \quad \bar{y} = \sum_{i=1}^{9} \bar{y}_i N_i \tag{2.2}$$

For the 2-D case, the Jacobian is given by a matrix:

$$\boldsymbol{J} = \begin{bmatrix} {}^{\partial x}\!/_{\partial \xi} & {}^{\partial y}\!/_{\partial \xi} \\ {}^{\partial x}\!/_{\partial \eta} & {}^{\partial y}\!/_{\partial \eta} \end{bmatrix} \tag{2.3}$$

Whereas the derivatives of the shape functions needed to calculate the Jacobian are:

$$\begin{aligned}
\frac{\partial N_1}{\partial \xi} &= \frac{1}{4}\eta(2\xi-1)(\eta-1) & \frac{\partial N_1}{\partial \eta} &= \frac{1}{4}\xi(\xi-1)(2\eta-1) \\
\frac{\partial N_2}{\partial \xi} &= \frac{1}{4}\eta(2\xi+1)(\eta-1) & \frac{\partial N_2}{\partial \eta} &= \frac{1}{4}\xi(\xi+1)(2\eta-1) \\
\frac{\partial N_3}{\partial \xi} &= \frac{1}{4}\eta(2\xi+1)(\eta+1) & \frac{\partial N_3}{\partial \eta} &= \frac{1}{4}\xi(\xi+1)(2\eta+1) \\
\frac{\partial N_4}{\partial \xi} &= \frac{1}{4}\eta(2\xi-1)(\eta+1) & \frac{\partial N_4}{\partial \eta} &= \frac{1}{4}\xi(\xi-1)(2\eta+1) \\
\frac{\partial N_5}{\partial \xi} &= -\xi\eta(\eta-1) & \frac{\partial N_5}{\partial \eta} &= \frac{1}{2}(1-\xi^2)(2\eta-1) \\
\frac{\partial N_6}{\partial \xi} &= \frac{1}{2}(2\xi+1)(1-\eta^2) & \frac{\partial N_6}{\partial \eta} &= -\xi\eta(\xi+1) \\
\frac{\partial N_7}{\partial \xi} &= -\xi\eta(\eta+1) & \frac{\partial N_7}{\partial \eta} &= \frac{1}{2}(1-\xi^2)(2\eta+1) \\
\frac{\partial N_8}{\partial \xi} &= \frac{1}{2}(2\xi-1)(1-\eta^2) & \frac{\partial N_8}{\partial \eta} &= -\xi\eta(\xi-1) \\
\frac{\partial N_9}{\partial \xi} &= -2\xi(1-\eta^2) & \frac{\partial N_9}{\partial \eta} &= -2\eta(1-\xi^2)
\end{aligned} \tag{2.4}$$

The relations stated on Equation 2.4 allow the evaluation of the Jacobian at the node 2, with coordinates $(\xi, \eta) = (1, -1)$, yielding:

$$\boldsymbol{J}_{\text{node 2}} = \begin{bmatrix} \frac{l}{2} - 2\alpha l & 0 \\ 0 & \frac{l}{2} \end{bmatrix} \tag{2.5}$$

For the 2-D case, the singularity takes place when the determinant of the Jacobian is zero. That is, the matrix cannot be inverted (analogous to the division by zero in the 1-D case). The determinant of the Jacobian takes the value of zero for:

$$|\boldsymbol{J}| = \frac{l^2}{4} - l^2\alpha = 0 \quad \Rightarrow \quad \alpha = \frac{1}{4} \tag{2.6}$$

We notice that, again, $\alpha = \frac{1}{4}$ implicates on a singular problem, where the strain tend to infinity and fracture mechanics may apply.