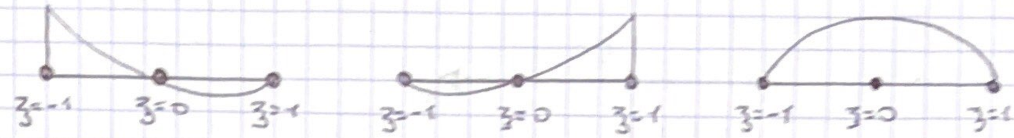
Assignment 5 - Student : Marcello Rubino

- **5.1**

•)



$N_1(\xi) = a_0 + a_1\xi + a_2\xi^2$

$N_2(\xi) = b_0 + b_1\xi + b_2\xi^2$

$N_3(\xi) = c_0 + c_1\xi + c_2\xi^2$

We impose the boundary conditions

$$\begin{cases} N_1(0) = 0 \rightarrow \bullet\, a_0 = 0 \\ N_1(-1) = 1 \rightarrow -a_1 + a_2 = 1 \rightarrow 2a_2 = 1 \rightarrow \bullet\, a_2 = \frac{1}{2} \\ N_1(1) = 0 \rightarrow a_1 + a_2 = 0 \rightarrow a_1 = -a_2 \rightarrow \bullet\, a_1 = -\frac{1}{2} \end{cases}$$

$$\begin{cases} N_2(0) = 0 \rightarrow \bullet\, b_0 = 0 \\ N_2(-1) = 0 \rightarrow -b_1 + b_2 = 0 \rightarrow b_1 = b_2 \rightarrow \bullet\, b_1 = \frac{1}{2} \\ N_2(1) = 1 \rightarrow b_1 + b_2 = 1 \rightarrow 2b_2 = 1 \rightarrow \bullet\, b_2 = \frac{1}{2} \end{cases}$$

$$\begin{cases} N_3(0) = 1 \rightarrow \bullet\, c_0 = 1 \\ N_3(-1) = 0 \rightarrow 1 - c_1 + c_2 = 0 \rightarrow c_1 = 1 + c_2 \rightarrow \bullet\, c_1 = 1 - 1 = \emptyset \\ N_3(1) = 0 \rightarrow 1 + c_1 + c_2 = 0 \rightarrow 1 + 1 + 2c_2 \rightarrow 2c_2 = -2 \rightarrow \bullet\, c_2 = -1 \end{cases}$$

$\downarrow$

$$N_1(\xi) = -\frac{\xi}{2} + \frac{\xi^2}{2}$$

$$N_2(\xi) = \frac{\xi}{2} + \frac{\xi^2}{2}$$

$$N_3(\xi) = 1 - \xi^2$$

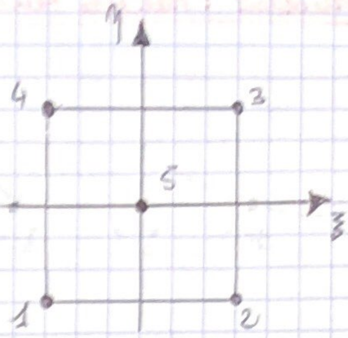
••) If I sum the shape functions, $\forall \xi \in [-1, 1]$,

$$N_1(\xi) + N_2(\xi) + N_3(\xi) = -\cancel{\frac{\xi}{2}} + \frac{\xi^2}{2} + \cancel{\frac{\xi}{2}} + \frac{\xi^2}{2} + 1 - \xi^2 = 1$$

•••) The first derivatives of the shape functions respect to the natural coordinates:

$$N_1'(\xi) = -\frac{1}{2} + \xi, \quad N_2'(\xi) = \frac{1}{2} + \xi, \quad N_3'(\xi) = -2\xi$$

• 5.2



using the line product method, the shape function linked to the node 5 is:

$$N_5(\xi,\eta) = C_5 \cdot L_{1-2} \cdot L_{2-3} \cdot L_{3-4} \cdot L_{1-4}$$

where L_{1-2} is, for example the equation of the line 1-2 where the shape function must be $= \emptyset$.

$$N_5(\xi,\eta) = C_5(\xi^2-1)(\eta^2-1)$$
$$N_5(0,0) = C_5 = 1 \rightarrow C_5 = 1$$
$$\downarrow$$
$$\underline{N_5(\xi,\eta) = (\xi^2-1)(\eta^2-1)}$$

Considering the 4-node quad, the shape functions are the following:

$$N_1^{IV}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2^{IV}(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3^{IV}(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4^{IV}(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Now, combining them with the N_5, we calculate the right coefficient α that makes them $= \emptyset$ if calculated on node 5

$$N_1(\xi,\eta) = N_1^{IV}(\xi,\eta) + \alpha N_5(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) + \alpha(\xi^2-1)(\eta^2-1)$$
$$N_1(0,0) = \frac{1}{4} + \alpha = 0 \longrightarrow \alpha = -\frac{1}{4}$$
$$N_2(\xi,\eta) = N_2^{IV}(\xi,\eta) + \alpha N_5(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta) + \alpha(\xi^2-1)(\eta^2-1)$$
$$N_2(0,0) = \frac{1}{4} + \alpha \cdot 0 \longrightarrow \alpha = -\frac{1}{4}$$
$$N_3(\xi,\eta) = N_3^{IV}(\xi,\eta) + \alpha N_5(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta) + \alpha(\xi^2-1)(\eta^2-1)$$

$$N_3(0,0) = \frac{1}{4} + \alpha = 0 \rightarrow \alpha = -\frac{1}{4}$$

$$N_4(\xi,\eta) = N_4^{N}(\xi,\eta) + \alpha N_5(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta) + \alpha(\xi^2-1)(\eta^2-1)$$

$$N_4(0,0) = \frac{1}{4} + \alpha = 0 \rightarrow \alpha = -\frac{1}{4}$$

So here there are the shape functions modified:

$$\begin{cases} N_1(\xi,\eta) = \frac{1}{4}\left[(1-\xi)(1-\eta) - (\xi^2-1)(\eta^2-1)\right] \\ N_2(\xi,\eta) = \frac{1}{4}\left[(1+\xi)(1-\eta) - (\xi^2-1)(\eta^2-1)\right] \\ N_3(\xi,\eta) = \frac{1}{4}\left[(1+\xi)(1+\eta) - (\xi^2-1)(\eta^2-1)\right] \\ N_4(\xi,\eta) = \frac{1}{4}\left[(1-\xi)(1+\eta) - (\xi^2-1)(\eta^2-1)\right] \\ N_5(\xi,\eta) = (\xi^2-1)(\eta^2-1) \end{cases}$$

Let's verify now that the sum of the shape functions is $=1$:

$$\sum = N_1+N_2+N_3+N_4+N_5 = \frac{1}{4}(1-\cancel{\xi}-\cancel{\eta}+\cancel{\xi\eta}) + \frac{1}{4}(1+\cancel{\xi}-\cancel{\eta}-\cancel{\xi\eta})$$

$$+\frac{1}{4}(1+\cancel{\xi}+\cancel{\eta}+\cancel{\xi\eta}) + \frac{1}{4}(1-\cancel{\xi}+\cancel{\eta}-\cancel{\xi\eta}) - \cancel{\frac{4}{4}(\xi^2-1)(\eta^2-1)} +$$

$$+\cancel{(\xi^2-1)(\eta^2-1)} = 1$$

• 5.3

1) n nodes $= 8 \rightarrow n_{DoF} = 3 \cdot n$ nodes $= 24$

$n_{DoF} - n_R = 24 - 6 = 18 \rightarrow n_E = 6$

(Rigid modes) (order of E matrix)

(minimum number of Gauss int. points) $n_{G_{MIN}} = \frac{n_{DoF} - n_R}{n_E} = 3$ INTEGRATION RULE $2\times2\times2$

2) n nodes $= 20 \rightarrow n_{DoF} = 60 \rightarrow n_{DoF} - n_R = 60 - 6 = 54$

$n_E = 6 \rightarrow n_{G_{MIN}} = \frac{n_{DoF} - n_R}{n_E} = 9$ INTEGRATION RULE $3\times3\times3$

3) n nodes $= 27 \rightarrow n_{DoF} = 81 \rightarrow n_{DoF} - n_R = 81 - 6 = 75$

$n_E = 6 \rightarrow n_{G_{MIN}} = \frac{n_{DoF} - n_R}{n_E} = 12.5 \rightarrow 13$ INTEGRATION RULE $3\times3\times3$

4) n nodes $= 64 \rightarrow n_{DoF} = 192 \rightarrow n_{DoF} - n_R = 192 - 6 = 186$

$n_E = 6 \rightarrow n_{G_{MIN}} = \frac{n_{DoF} - n_R}{n_E} = 31$ INTEGRATION RULE : $4\times4\times4$