Contents

Assignment 5.1

The isoparametric definition of the straight-node bar element in its local system x is

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

In which N^e are the shape functions of the element and ξ is the isoparametric coordinate that takes values -1, 1 and 0 at nodes 1, 2 and 3.

The coordinates of the nodes are displayed as follows

$$\bar{x}_1 = 0 \qquad \bar{x}_2 = l \qquad \bar{x}_3 = \left(\frac{1}{2} + \alpha\right) l$$

<u>Study of the Jacobian matrix</u>

First of all, the shape functions are described as follows

$$N_1^e = \frac{\xi}{2}(\xi - 1)$$
$$N_2^e = \frac{\xi}{2}(\xi + 1)$$
$$N_3^e = (1 - \xi)(1 + \xi)$$

From the definition of the isoparametric element the local coordinate can be expressed as a function of ξ

$$\bar{x} = \sum_{i=1}^{3} \bar{x}_i N_i(\xi)$$

And, therefore, the Jacobian matrix is defined as

$$J = \frac{d\bar{x}}{d\xi} = \sum_{i=1}^{3} \bar{x}_i \frac{dN_i(\xi)}{d\xi}$$

$$J = l\left(\frac{1}{2} - 2\alpha\xi\right)$$

Then, in order to find the minimum $|\alpha|$ value for which the Jacobian vanishes, the previous expression is forced to zero.

$$\alpha = \frac{1}{4}\frac{1}{\xi}$$

in absolute terms

$$|\alpha| = \frac{1}{4}\left|\frac{1}{\xi}\right|$$

Therefore, the minimum value of the parameter $|\alpha|$ coincides with the maximum absolute value of the isoparametric coordinate ξ.

$$|\alpha| = \left|\frac{1}{4}\right|$$

<u>Effects on the axial strain</u>

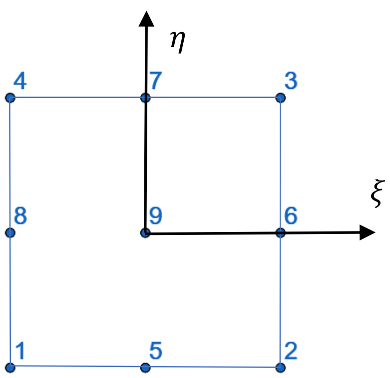
The expression of the axial strain in 1D is

$$\varepsilon = \frac{d\bar{u}}{d\bar{x}} = \frac{d\bar{u}}{d\xi}\frac{d\xi}{d\bar{x}} = J^{-1}\frac{d\bar{u}}{d\xi} = \frac{1}{l\left(\frac{1}{2} - 2\alpha\xi\right)}\frac{d\bar{u}}{d\xi}$$

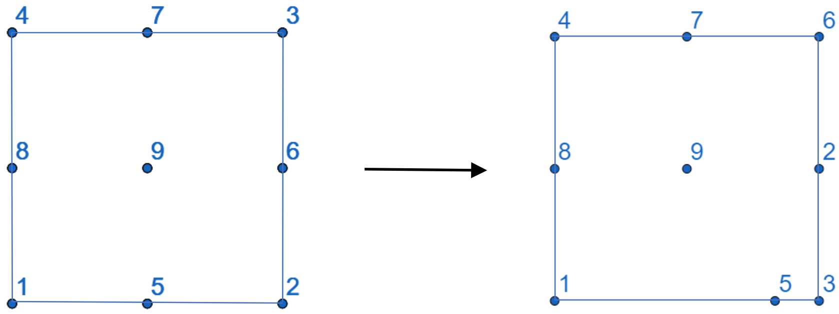
It is obvious that when α takes values $\pm 1/4$ the Jacobian matrix is null, and thus, the inverse becomes unbounded (it tends to infinite) at the end points ($\xi = \pm 1$)

Assignment 5.2

The element is initially a perfect square, nodes 5,6,7,8 are the midpoint of the sides 1-2, 2-3, 3-4 and 4-1, respectively, and 9 is the centre of the square.



Study of the Jacobian determinant at node 2 moving node 5 horizontally



The shape functions for an isoparametric formulation of the above element are

$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta \qquad N_5^e = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_2^e = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta \qquad N_6^e = \frac{1}{2}(1+\xi)(1-\eta^2)\xi$$

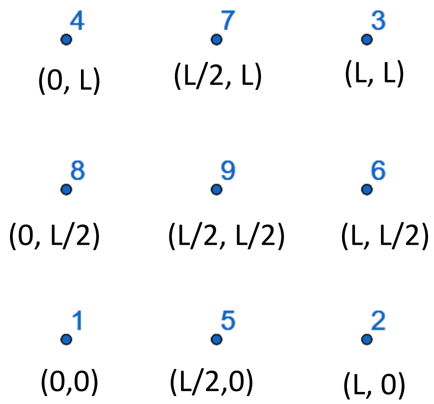
$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta)\xi\eta \qquad N_7^e = \frac{1}{2}(1-\xi^2)(1+\eta)\eta$$

$$N_4^e = -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta \qquad N_8^e = -\frac{1}{2}(1-\xi)(1-\eta^2)\xi$$

$$N_9^e = (1-\xi^2)(1-\eta^2)$$

For the isoparametric coordinates from -1 to 1

The initial Cartesian node coordinates are



Setting a parameter α to describe the horizontal movement of node 5, its x-coordinate results as

$$\bar{x}_5 = \frac{l}{2} + \alpha$$

We can interpolate the geometry between the nodes by using shape functions as follows

$$x = \sum_{i=1}^{9} x_i N_i(\xi, \eta) \qquad , \qquad y = \sum_{i=1}^{9} y_i N_i(\xi, \eta)$$

The Jacobian matrix in this 2D problem is described as

$$J = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix}$$

By using the symbolic toolbox of Matlab the following parameters have been calculated

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (\xi + 1)(\eta^2\alpha - \eta\alpha + \xi\eta\alpha - \xi\eta^2\alpha + 1) \\ \eta + 1 \end{bmatrix}$$

$$J = \frac{1}{2} \begin{bmatrix} 2\alpha\xi\eta - \alpha\xi\eta^2 + L & 0 \\ -\alpha(2\eta - 1)(\xi^2 - 1) & L \end{bmatrix}$$

$$|J| = \frac{L}{4}(2\alpha\xi\eta - 2\alpha\xi\eta^2 + L)$$

- Initial case ($\alpha = 0$)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \xi + 1 \\ \eta + 1 \end{bmatrix}$$

$$J = \frac{1}{2} \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$$

$$|J| = \frac{L^2}{4}$$

Forcing the Jacobian expression to zero at node 2 $(\xi, \eta) = (-1,1)$ the bounding α parameter is

$$\alpha = \frac{L}{4}$$

Therefore, when the node 5 reaches the third quarter of the side 1-2 the Jacobian determinant vanishes at node 2 and it indicates a singularity at this point.

This result means that for this kind of element, the nodes located on the edges must be located between the first and the third quarter to avoid singularities, as it has been shown previously in the 1D case.

Appendix

<u>Matlab code</u>

```
syms L alpha xi nu

%% initial case (alpha=0)
alpha=0;

%% alpha case
x=[0 L L 0 L/2+alpha L L/2 0 L/2];
y=[0 0 L L 0 L/2 L L/2 L/2];

N=[1/4*(1-xi)*(1-nu)*xi*nu;
-1/4*(1+xi)*(1-nu)*xi*nu;
1/4*(1+xi)*(1+nu)*xi*nu;
-1/4*(1-xi)*(1+nu)*xi*nu;
-1/2*(1-xi^2)*(1-nu)*nu;
1/2*(1+xi)*(1-nu^2)*xi;
1/2*(1-xi^2)*(1+nu)*nu;
-1/2*(1-xi)*(1-nu^2)*xi;
(1-xi^2)*(1-nu^2)];

N=transpose(N);

dN_dxi=(diff(N,xi));
dN_dnu=(diff(N,nu));

X=(sum((x.*N)))
Y=(sum((y.*N)))

dx_dxi = (sum(x.*dN_dxi));
dx_dnu = (sum(x.*dN_dnu));
dy_dxi = (sum(y.*dN_dxi));
dy_dnu = (sum(y.*dN_dnu));

J=[dx_dxi dy_dxi;
    dx_dnu dy_dnu]

detJ=det(J)
```