Assignment 5

Nadim Saridar

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Assignment 5.1

This exercise can be represented like in the figure below:

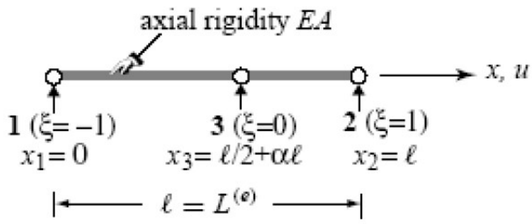

Figure 1: Representation of the Bar Element

This is a 3 noded 1D element, therefore the shape functions are the following:

$$N_1 = \frac{\xi(\xi-1)}{2}$$
$$N_2 = \frac{\xi(\xi+1)}{2}$$
$$N_3 = 1-\xi^2$$

From the slides of lecture 6, the equation to calculate the jacobian for a 1D element is the following:

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{3} x_i \frac{\partial N_i}{\partial \xi}$$

From equation(7.1), and replacing x_i and N_i by their corresponding values, we get:

$$x = \frac{\xi(\xi+1)}{2} l + \left(\frac{l}{2} + \alpha l\right)(1-\xi^2)$$

Therefore, the jacobian is the following:

$$J = \frac{2\xi+1}{2} l - \xi l - 2\xi\alpha l$$

$$J = \frac{l - 4\xi\alpha}{2}$$

Placing this equation bigger or equal to zero ($J \geq 0$), two cases are taken: $\xi = 1$ and $\xi = -1$. l is always positive, therefore it is removed from the equation. Putting both cases together, the following is obtained:

$$\frac{-1}{4} \leq \alpha \leq \frac{1}{4}$$

Placing $\alpha = 0$, we get $J = 0$. Therefore to calculate the stifness matrix, one should get the strain displacement matrix $\boldsymbol{B}$.

$$\boldsymbol{B} = J^{-1} \frac{d\boldsymbol{N}}{d\xi}$$

Therefore the inverse of the jacobian tends to infinity ($J^{-1} = \frac{2}{0}$).

Assignment 5.2

The element described in this exercise is a 9 node quadralatiral element, like in the figure below:

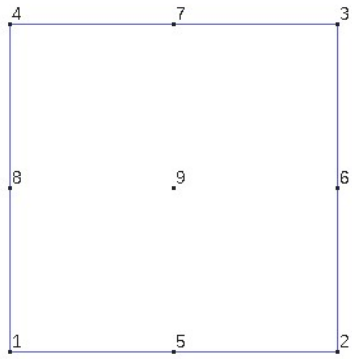

Figure 2: Representation of the 9 Node Quadralatiral Element

Writing this problem in a isoparametric representation:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \end{bmatrix}$$

The shape functions are the ones for a quadratic quadrilatiral element in the reference element. The jacobian in this 2D case is a 2×2 matrix:

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Therefore, the jacobian can be written as following:

$$\begin{aligned} J(1,1) &= \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \xi} \\ J(1,2) &= \sum_{i=1}^{9} x_i \frac{\partial N_i}{\partial \eta} \\ J(2,1) &= \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \xi} \\ J(2,2) &= \sum_{i=1}^{9} y_i \frac{\partial N_i}{\partial \eta} \end{aligned}$$

The determinant of the jacobian is the following:

$$|J| = J(1,1)J(2,2) - J(2,1)J(1,2)$$

After calculating each element of the jacobian matrix, we notice that $J(2,1) = 0$ therefore for calculating the determinant, $J(1,1)J(2,2$ is enough to be calculated.

Choosing the coordinates of the points similar to the first exercise, and following the given of this problem that node 5 should get closer to node 2:

$$\begin{aligned} &x_1 = x_4 = x_8 = 0 \qquad && y_1 = y_2 = y_5 = 0 \\ &x_7 = x_9 = L/2 && y_6 = y_8 = y_9 = L/2 \\ &x_2 = x_3 = x_6 = L && y_3 = y_4 = y_7 = L \\ &x_5 = (L/2 + \alpha L) \quad (\alpha \geq 0) \end{aligned}$$

To solve for alpha, only $J(1,1)$ needs to be calculated and then solved for the value of α to get $|J| = 0$.

After solving for α, the same value from the first exercise is obtained $\alpha = 1/4$.

Therefore one can conclude that if a quadratic element is used, one should make sure that a middle node (in the case of this exercise nodes 5,6,7 and 8) should not be in a distance of $1/4^{th}$ of the the length of the element's side to a side node (in the case of this exercise nodes 1,2,3 and 4).

Below is the code used to solve this exercise:

```
Exercise2.m*
syms chi eta L alpha
% Defining the shape functions and the derivative %
% with respect to chi.                            %
N1 = 1/4*(1-chi)*(1-eta)*chi*eta;
N1c=diff(N1,chi);
N2 = -1/4*(1+chi)*(1-eta)*chi*eta;
N2c=diff(N2,chi);
N3 = 1/4*(1+chi)*(1+eta)*chi*eta;
N3c=diff(N3,chi);
N4 = -1/4*(1-chi)*(1+eta)*chi*eta;
N4c=diff(N4,chi);
N5 = -1/2*(1-chi^2)*(1-eta)*eta;
N5c=diff(N5,chi);
N6 = 1/2*(1+chi)*(1-eta^2)*chi;
N6c=diff(N6,chi);
N7 = 1/2*(1-chi^2)*(1+eta)*eta;
N7c=diff(N7,chi);
N8 = -1/2*(1-chi)*(1-eta^2)*chi;
N8c=diff(N8,chi);
N9 = (1-chi^2)*(1-eta^2);
N9c=diff(N9,chi);
% Calculating J(2,1) %
y1 = (N3c+N4c+N7c)*L;
y2 = (N6c+N8c+N9c)*(L/2);
Jy = y1+y2;
Jy = simplify(Jy);
disp('Jy =')
disp(Jy);
% Calculating J(1,1) %
x1 = (N2c+N3c+N6c)*L;
x2 = (N7c+N9c)*(L/2);
x3 = (L/2+alpha*L)*(N5c);
J=x1+x2+x3;
simplify(J);
chi=1;eta=-1;

J=subs(J);
% Solving for alpha %
alpha=solve(J);
disp('alpha=');
disp(alpha);
```

Figure 3: Script for Exercise2

```
>> Exercise2
Jy =
0

alpha=
1/4
```

Figure 4: Results for Exercise2