Computational Structural Mechanics and Dynamics - Homework 5

Martin Vee Akselsen - Erasmus student

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1 Introduction

This report describes the solution of Assignment 5 in the subject Computational Structural Mechanics and Dynamics.

Contents

2 Exercise 1

$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2 \quad N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2 \quad N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2$

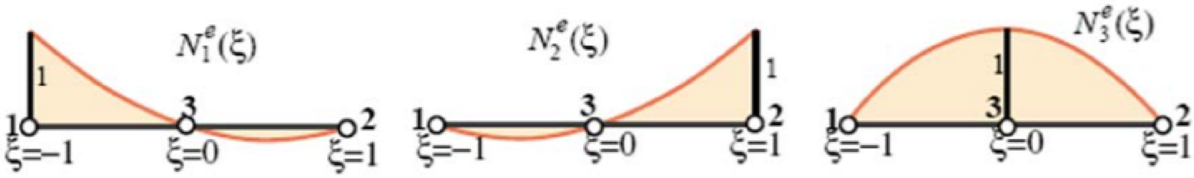


Figure.- Isoparametric shape functions for 3-node bar element (sketch).
Node 3 has been drawn at the 1-2 midpoint but it may be moved away from it.

2.1

$$N_1(\xi) = a_o + a_1\xi + a_2\xi^2 \tag{1}$$

I: $N_1(-1) = a_0 - a_1 + a_2 = 1$
II: $N_1(0) = a_0 = 0$
III: $N_1(1) = a_1 + a_2 = 0 \rightarrow a_1 = -a_2$

I: $2a_2 = 1 \rightarrow a_2 = \frac{1}{2}$
III: $a_1 = -\frac{1}{2}$

$$N_1(\xi) = -\frac{1}{2}\xi + \frac{1}{2}\xi^2 \tag{2}$$

$$N_2(\xi) = b_o + b_1\xi + b_2\xi^2 \tag{3}$$

I: $N_2(-1) = b_0 - b_1 + b_2 = 0$
II: $N_2(0) = b_0 = 0$
III: $N_2(1) = b_1 + b_2 = 1 \rightarrow b_1 = 1 - b_2$

I: $b_2 = b_1$
III: $b_1 = \frac{1}{2} \rightarrow b_2 = \frac{1}{2}$

$$N_2(\xi) = \frac{1}{2}\xi + \frac{1}{2}\xi^2 \tag{4}$$

$$N_3(\xi) = c_o + c_1\xi + c_2\xi^2 \tag{5}$$

I: $N_3(-1) = c_0 - c_1 + c_2 = 0$
II: $N_2(0) = c_0 = 1$
III: $N_2(1) = c_0 + c_1 + c_2 = 0 \rightarrow c_1 = -1 - c_2$

I: $c_2 = -1 + c_1 \rightarrow c_2 = -1 - 1 - c_2 \rightarrow c_2 = -1$
III: $c_1 = 0$

$$N_3(\xi) = 1 - \xi^2 \tag{6}$$

2.2

The sum of the three shape functions:

$$N_1 + N_2 + N_3 = -\tfrac{1}{2}\xi + \tfrac{1}{2}\xi^2 + \tfrac{1}{2}\xi + \tfrac{1}{2}\xi^2 + 1 - \xi^2 = 1$$

We could also easily see that that the sum of the shape-functions in the points also is equal to zero.

2.3

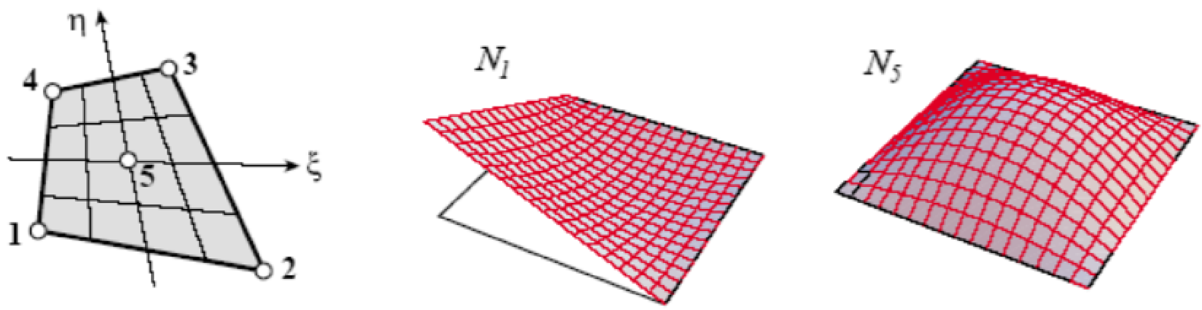
The derivative of the shape functions in respect to the natural coordinates.

$$\frac{dN_1}{d\xi} = -\frac{1}{2} + \xi \tag{7}$$

$$\frac{dN_2}{d\xi} = \frac{1}{2} + \xi \tag{8}$$

$$\frac{dN_3}{d\xi} = 2\xi \tag{9}$$

3 Exercise 2



With the use of the line-product method we could develop the shape function N_5. Hence, considering that we have to cover all points of the quadrilateral element except node 5, we could writhe N_5 as:

$$N_5 = c_5 L_{2-3} L_{3-4} L_{4-1} = c_5(\xi - 1)(\eta - 1)(\xi + 1) = c_5(1 - \xi^2)(1 - \eta)$$

Using the normalization condition in point (0,0) we obtain $c_5 = 1$.

$$N_5 = (1 - \xi^2)(1 - \eta) \tag{10}$$

To calculate the shape rest of the shape functions we use the shape functions a 4-noded reactangle and combine it with the all ready obtained N_5 to ensure that the rest of the shape functions take 0 in node 5 using the expression:

$$N_i = N_i^* + \alpha N_5, \text{ i} = 1,2,3,4$$

From the course slides CSMD6-Isoparametric we have that the shape functions for a 4-noded rectangle is:

$N_1^* = \frac{1}{4}(1-\xi)(1-\eta)$
$N_2^* = \frac{1}{4}(1+\xi)(1-\eta)$
$N_3^* = \frac{1}{4}(1+\xi)(1+\eta)$
$N_4^* = \frac{1}{4}(1-\xi)(1+\eta)$

To determine α we use the condition that N_i vanishes in point (0,0) giving us.

$$N_i(0,0) = \frac{1}{4} + \alpha = 0 \rightarrow \alpha = -\frac{1}{4}$$

Giving us the rest of the shape functions.

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) \tag{11}$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) \tag{12}$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) \tag{13}$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) \tag{14}$$

Again we check if the sum of the shape functions is equal to 1.

$$N_1+N_2+N_3+N_4+N_5 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) + \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) +$$
$$+\frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) + \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) + (1-\xi^2)(1-\eta) = 1$$

Where we could easily see that the the values for the shape function 5 gets eliminated by the additional part for the first four shape functions. At the same time we know that the sum of the four shape functions for a 4-noded rectangle is equal to 1. Hence, also here we have that the sum of the five shape functions is equal to 1.

4 Exercise 3

To attain rank sufficiency - $n_E n_G$ must be equal or greater than $n_F - n_R$

$$n_E n_G = n_F - n_R \tag{15}$$

Where n_E is the order of the stress strain matrix $\mathbf{E}$ and n_G is the number of Gauss points when integrated numerically. While n_F is the number of degrees of freedom and n_R is the number of independent rigid body modes.

For the hexahedron the n_E is equal to 6 meaning that we add 6 points to the stiffness matrix for each Gauss point. The number of independent rigid body modes, n_F, is equal to 6. The number of degrees of freedom depends on how many nodes the hexahedron has.

	8-node Hex	**20-node Hex**	**27-node Hex**	**64-node Hex**
n_R	18	60	91	192

Using Equation 15 we find the number of Gauss points needed to attain a rank sufficient stiffness matrix.

	8-node Hex	**20-node Hex**	**27-node Hex**	**64-node Hex**
n_G	3	9	10	31