International Centre for
Numerical Methods in Engineering

Universitat Politècnica de Catalunya

Master of Science in Computational Mechanics

Computational Structural Mechanics and Dynamics

Assignment 6

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Submitted To:
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ASSIGNMENT 6

a. *Program In MatLab the Timoshenko 2 Nodes Beam element with reduce integration for the shear stiffness matrix*

b. *Solve the following problem with a 64 element mesh with the:*

- 2 nodes Euler Bernulli element.
- 2 nodes Timoshenko Full Integrate element.
- 2 nodes Timoshenko Reduce Integration element.

Compare maximum displacements, moments and shear for the 3 elements against the a/L relationship, the dimensions considered are listed next:

1. $a = 0,001$
2. $a = 0,005$
3. $a = 0,010$
4. $a = 0,020$
5. $a = 0,050$
6. $a = 0,100$
7. $a = 0,200$
8. $a = 0,400$

The geometry of the beam to analyze, its load and mechanical properties are depicted in the figure 0.1

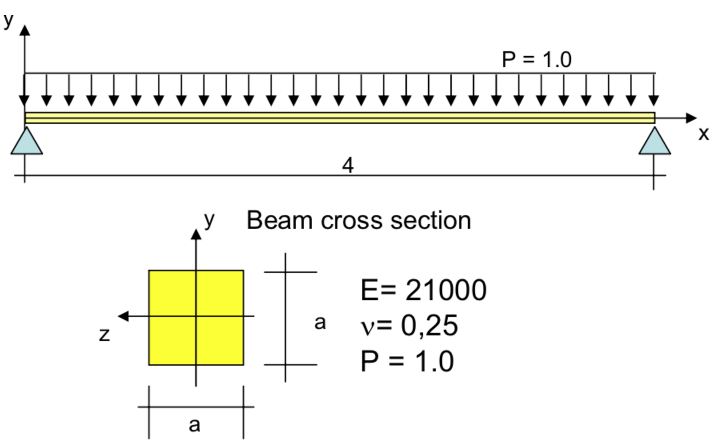

Figure 0.1: Geometry of the beam, load and mechanical properties.

Solution:

As seen in theory, the Timoshenko formulation is based in the sum of two matrices which represents the bending effect and the shear contribution[1]:

$$\left[\mathbf{K}_b^{(e)} + \mathbf{K}_s^{(e)}\right] (\mathbf{a})^{(\mathbf{e})} = \mathbf{f} \tag{0.1}$$

The exact evaluation of the bending stiffness matrix $\mathbf{K}_b^{(e)}$ requires a single Gauss integration point, as all the terms are constant. Meanwhile the exact integration of the shear stiffness matrix $\mathbf{K}_s^{(e)}$ requires two Gauss points, as quadratic terms are involved. Moreover, the Timoshenko formulation can be useful when the beam is thick, which results are adequate. In contrast, a test can be done by increasing the slenderness of a beam and found that Timoshenko formulation is progressively stiffer than the exact one. This phenomenon is known as *shear locking*. Then, a different approach to avoid this problem is by "under-integrating" the terms of the shear stiffness matrix using an order less than is needed for the exact integration, this methodology is named **Reduced Integration**.

Then, the implementation in the code is similar to the one used for Timoshenko theory, but the stiffness matrices are modified by using a single integration point, which leads to the next expressions:

$$\mathbf{K}_b^{(e)} = \frac{EI}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{0.2}$$

$$\mathbf{K}_s^{(e)} = \frac{GA^*}{l} \begin{bmatrix} 1 & \frac{l}{2} & -1 & \frac{l}{2} \\ & \frac{l^2}{4} & -\frac{l}{2} & \frac{l^2}{4} \\ & & 1 & -\frac{l}{2} \\ SYM & & & \frac{l^2}{4} \end{bmatrix} \tag{0.3}$$

where:

- E is the Young Modulus.
- I is the Inertia of the cross section.
- l is the length of the element.
- G is the Shear Modulus.
- A^* is the reduced area by the shape shear coefficient.

[1] Which requires a parameter depending the shape of the cross section of the beam

The properties considered in the computation are listed in the next table:

Test	Dimension [m]	Area [m2]	Inertia [m4]
1	0.001	0.000001	8.33333E-14
2	0.005	0.000025	5.20833E-11
3	0.01	0.0001	8.33333E-10
4	0.02	0.0004	1.33333E-08
5	0.05	0.0025	5.20833E-07
6	0.1	0.01	8.33333E-06
7	0.2	0.04	0.000133333
8	0.4	0.16	0.002133333

Table 0.1: Dimension and area properties for each test.

Then, the maximum responses for displacement, moment and shear in absolute values, obtained by computing the three different formulations are listed in the next table:

Slenderness Tests	Euler Bernoulli			Full Timoshenko			Reduced Timoshenko		
a/L	Max Disp [m]	Max Mom [N m]	Max Shear [N]	Max Disp [m]	Max Mom [N m]	Max Shear [N]	Max Disp [m]	Max Mom [N m]	Max Shear [N]
2.50E-07	1.90E+09	1.9999	2	1.46E+06	0.0015	1.97	1.90E+09	1.999	1.97
6.25E-06	3.05E+06	1.9999	2	5.74E+04	0.0377	1.9687	3.05E+06	1.999	1.9688
2.50E-05	1.90E+05	1.9999	2	1.36E+04	0.1426	1.9688	1.90E+05	1.999	1.9687
1.00E-04	1.19E+04	1.9999	2	2.80E+03	0.4698	1.9688	1.19E+04	1.999	1.9687
6.25E-04	304.7621	1.9999	2	200.4275	1.3144	1.9688	304.7573	1.999	1.9687
0.0025	19.0476	1.9999	2	16.8752	1.7687	1.9687	19.0688	1.999	1.9688
0.01	1.1905	1.9999	2	1.1596	1.936	1.9688	1.1972	1.999	1.9688
0.04	0.0744	1.9999	2	0.0756	1.9829	1.9688	0.0762	1.999	1.9687

Table 0.2: Computation results using each methodology.

The above results are graphed in figures 0.2, 0.3 and 0.4. As it can be seen, the displacement, moment and shear progression between the Euler-Bernoulli and the Reduced Timoshenko are practically equal, in which it can be seen that the results have the same order and the magnitude is close each other. Meanwhile, the Timoshenko formulation using the full integration shows that for slender beams, which are the first tests, are quite different in order and magnitude. This is self-explanatory in the first paragraph of the assignment, where it was discussed that the complete integration add a shear phenomena that includes an over-stiffened values for *slender* beams. As the values of the slenderness considered as a/L increments, the approximation of Full Timoshenko is getting better as can be noticeable in the Moment graph, this proves that this expression works fine with *thick* beams. Finally as a conclusion of the Reduced Timoshenko is that this formulation performs much better as working with both types of beams than the original one, the approximation of the displacement, moment and shear is quite adequate. Which is interesting because the origin of the methodology was by using only one Gauss point integration instead of two, and this provides better solutions.

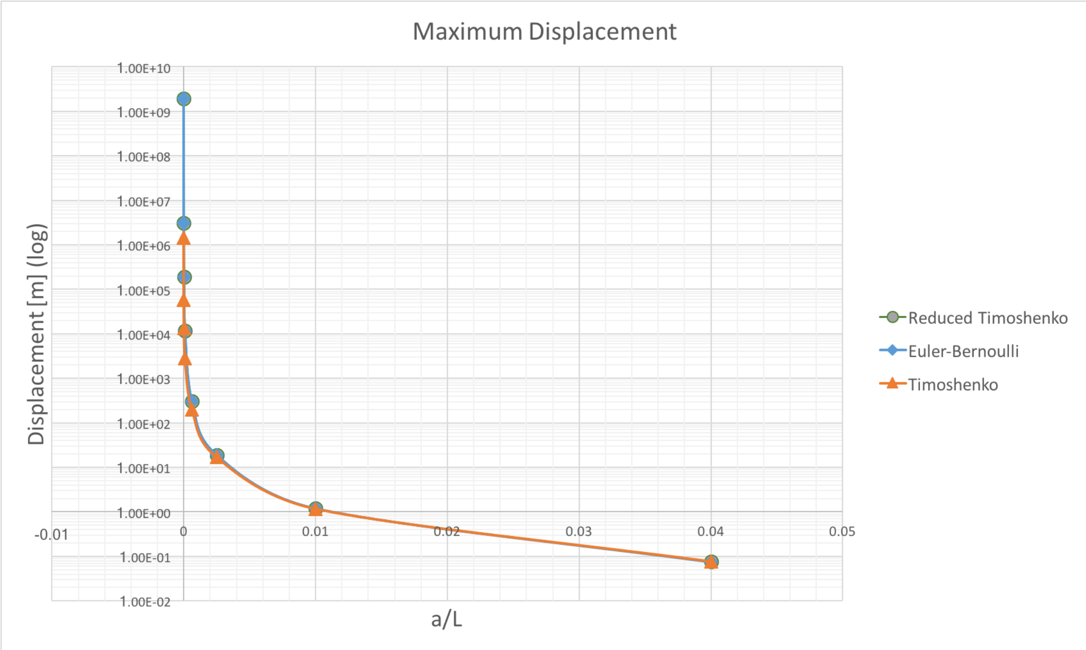

Figure 0.2: Displacement comparison between the EB, Timoshenko and Reduced Timoshenko.

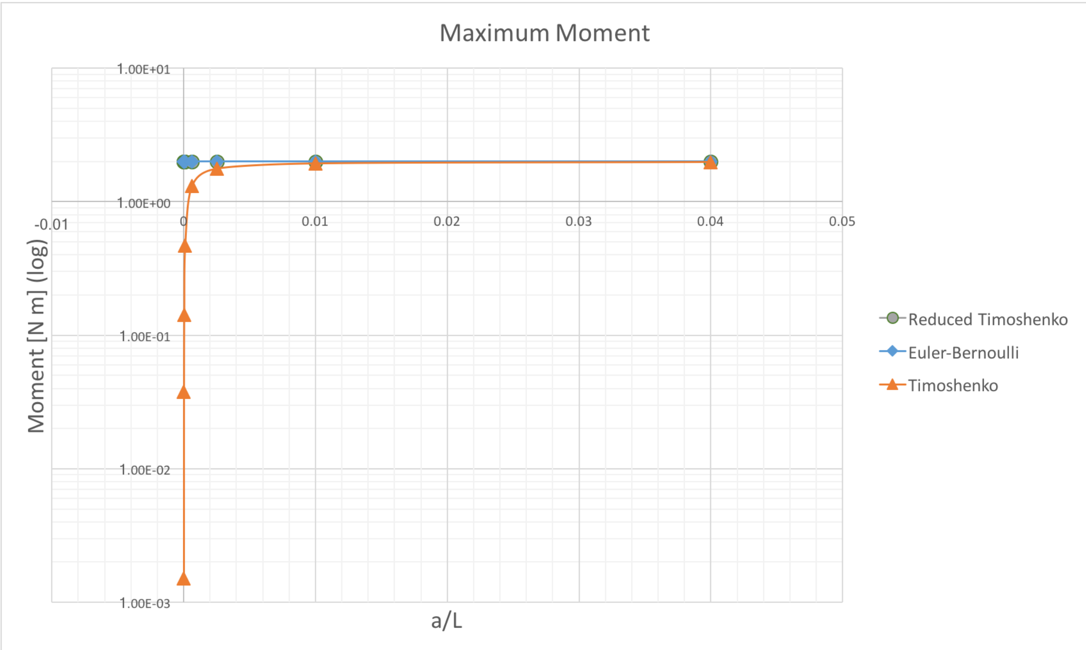

Figure 0.3: Moment comparison between the EB, Timoshenko and Reduced Timoshenko.

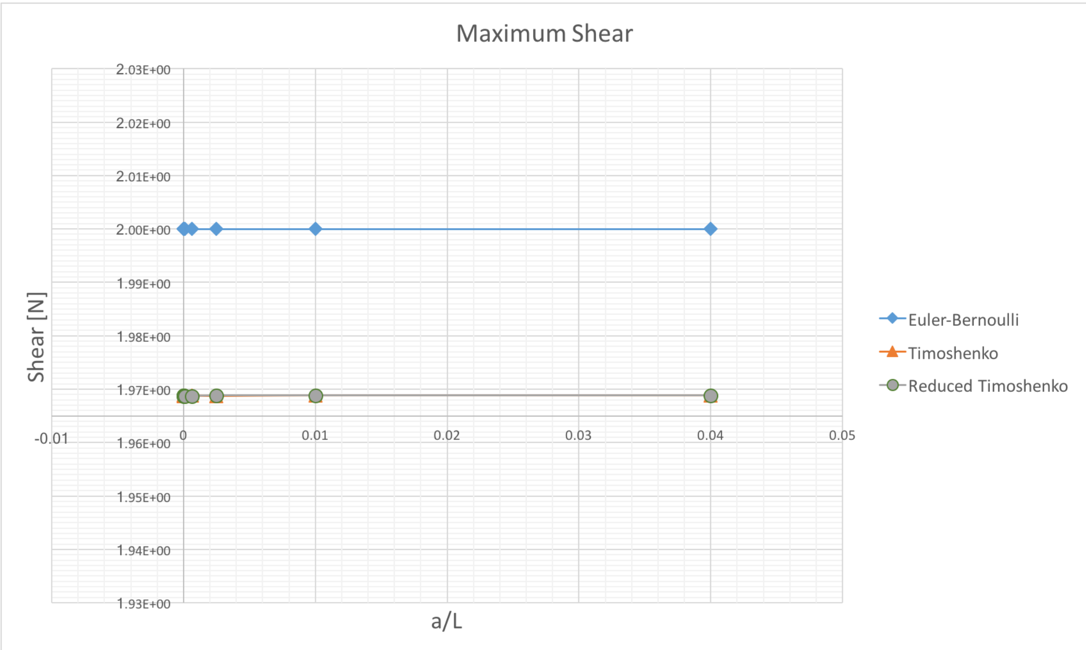

Figure 0.4: Moment comparison between the EB, Timoshenko and Reduced Timoshenko.