

### Assignment 3.1

(1)

a)

$$\frac{E}{(1-\nu)(1+\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \Rightarrow \frac{E}{(1-\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \frac{\nu}{1+\nu} & \frac{1}{1+\nu} & 0 \\ 0 & 0 & \frac{1}{2} \left( \frac{1-\nu}{1+\nu} \right) \end{bmatrix}$$

$$\text{Let } \nu^* = \frac{\nu}{1+\nu} \Rightarrow \nu = \frac{\nu^*}{1-\nu^*}$$

$$\frac{E}{1-\nu} \begin{bmatrix} \frac{\nu^*}{\nu} & \nu^* & 0 \\ \nu^* & \frac{\nu^*}{\nu} & 0 \\ 0 & 0 & \frac{1}{2} \left( \frac{1-\nu}{1+\nu} \right) \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \frac{E}{1 - \frac{\nu^*}{1-\nu^*}} &= \frac{E(1-\nu^*)}{1-2\nu^*} \\ \Rightarrow \frac{\nu^*}{\nu} &= \frac{\nu^*}{\frac{\nu^*}{1-\nu^*}} = 1-\nu^* \\ \Rightarrow \frac{1-\nu}{1+\nu} &= \frac{1 - \frac{\nu^*}{1-\nu^*}}{1 + \frac{\nu^*}{1-\nu^*}} = 1-2\nu^* \end{aligned}$$

$$\frac{E(1-\nu^*)}{1-2\nu^*} \begin{bmatrix} 1-\nu^* & \nu^* & 0 \\ \nu^* & 1-\nu^* & 0 \\ 0 & 0 & 1-2\nu^* \end{bmatrix}$$

$$\frac{E(1-\nu^*)(1+\nu^*)}{(1+\nu^*)(1-2\nu^*)}$$

$$\begin{aligned} \text{Let } E^* &= E(1-\nu^{*2}) \\ &= E \left[ 1 - \left( \frac{\nu}{1+\nu} \right)^2 \right] \\ &= E \left[ 1 - \frac{(1+\nu)^2 - \nu^2}{(1+\nu)^2} \right] \\ &= E \left[ \frac{1+2\nu + \nu^2 - \nu^2}{(1+\nu)^2} \right] \\ &= E \left[ \frac{1+2\nu}{(1+\nu)^2} \right] \end{aligned}$$

$$\frac{E^*(1-v^*)}{(1+v^*)(1-2v^*)} = \frac{E}{1-v^2} \quad (1)$$

$$\frac{E^* v^*}{(1+v^*)(1-2v^*)} = \frac{E v}{1-v^2} \quad (2)$$

$$\frac{1-v^*}{v^*} = \frac{1}{v}$$

$$v = \frac{v^*}{1-v^*}$$

Substitute in (1)

$$\frac{E^*(1-v^*)}{(1+v^*)(1-2v^*)} = \frac{E}{1-\frac{v^{*2}}{(1-v^*)^2}}$$

$$\frac{E^*}{(1-v^*)(1+v^*)(1-2v^*)} = \frac{E}{\frac{(1-v^*)^2 - v^{*2}}{(1-v^*)^2}}$$

$$E = \frac{E^*}{(1-v^{*2})}$$

Assignment 3.2

②

a)

$$E = 2G(1+\nu)$$

$$\lambda = \frac{2G(1+\nu)\nu}{(1+\nu)(1-2\nu)}$$

$$\frac{1-2\nu}{\nu} = \frac{2G}{\lambda}$$

$$\frac{1}{\nu} - 2 = \frac{2G}{\lambda}$$

$$\nu = \frac{\lambda}{2(G+\lambda)}$$

$$E = 2G \left( 1 + \frac{\lambda}{2(G+\lambda)} \right)$$

$$= 2G \left( \frac{2G+3\lambda}{2(G+\lambda)} \right)$$

$$= \frac{G(2G+3\lambda)}{G+\lambda}$$

where  $\mu = G$

b)

\* Plane stress:

$$\frac{E}{1-\nu^2} \rightarrow \frac{\frac{G(2G+3\lambda)}{G+\lambda}}{1 - \frac{\lambda^2}{4(G+\lambda)^2}} = \frac{\frac{G(2G+3\lambda)}{G+\lambda}}{\frac{4(G+\lambda)^2 - \lambda^2}{4(G+\lambda)^2}} = \frac{4G(2G+3\lambda)(G+\lambda)}{4(G+\lambda)^2 - \lambda^2}$$

$$\Rightarrow \frac{4G(2G+3\lambda)(G+\lambda)}{4(G+\lambda)^2 - \lambda^2} \begin{bmatrix} 1 & \frac{\lambda}{2(G+\lambda)} & 0 \\ \frac{\lambda}{2(G+\lambda)} & 1 & 0 \\ 0 & 0 & \frac{2G+\lambda}{G+\lambda} \end{bmatrix}$$

\* Plane strain:

$$\frac{E}{(1+\nu)(1-2\nu)} \rightarrow \frac{E}{1-\nu-2\nu^2} \rightarrow \frac{\frac{G(2G+3\lambda)}{G+\lambda}}{1 - \frac{\lambda}{2(G+\lambda)} - \frac{2\lambda^2}{4(G+\lambda)^2}} = \frac{2G(2G+3\lambda)}{2(G+\lambda) - \lambda - \frac{\lambda^2}{G+\lambda}}$$

$$\Rightarrow \frac{2G(2G+3\lambda)}{2(G+\lambda) - \lambda - \frac{\lambda^2}{G+\lambda}} \begin{bmatrix} \frac{2G+\lambda}{2(G+\lambda)} & \frac{\lambda}{2(G+\lambda)} & 0 \\ \frac{\lambda}{2(G+\lambda)} & \frac{2G+\lambda}{2(G+\lambda)} & 0 \\ 0 & 0 & \frac{1}{2} \left( \frac{G}{G+\lambda} \right) \end{bmatrix}$$

$$\Rightarrow \frac{G(2G+3\lambda)}{2(G+\lambda)^2 \left( \frac{2G+3\lambda}{2(G+\lambda)} \right) \left( \frac{G}{G+\lambda} \right)} \begin{bmatrix} 2G+\lambda & \lambda & 0 \\ \lambda & 2G+\lambda & 0 \\ 0 & 0 & G \end{bmatrix}$$

c)

$$\Rightarrow \begin{bmatrix} 2G & 0 & 0 \\ 0 & 2G & 0 \\ 0 & 0 & G \end{bmatrix} = \begin{bmatrix} \lambda & \lambda & -G \\ \lambda + G & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d)

$$E_G = \begin{bmatrix} \frac{E}{1+\nu} & 0 & 0 \\ 0 & \frac{E}{1+\nu} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### Assignment 3.3:

a)

The stiffness matrices for both models were computed using MATLAB scripts as follows:

#### Linear triangle:

```
%Outputs
% A area , V volume , K k_matrix , B B_matrix , e strain , Si-j-k , ali-j-k
% bei-j-k , omi-j-k

clear all
clc;

%Material properties
syms E ;
nu = 0;

%Define plane stress or plane strain ( 0/1 activate/deactivate )
plane_stress = 1;
plane_strain = 0;

%Coordinates of the triangle
xi = 0; yi = 0;
xj = 1; yj = 0;
xk = 0; yk = 1;

%Thickness of the triangle
h = 1;

%Define V for the cases of plane stress or plane strain
if (plane_stress == 1)
    v = (E/(1-nu^2)) * [ 1 nu 0 ; nu 1 0 ; 0 0 (1-nu)/2 ];
end

if (plane_strain == 1)
    v = (E/((1+nu)*(1-2*nu)))* [ (1-nu) nu 0 ; nu (1-nu) 0 ; 0 0 (0.5-nu)];
end

%Calculate the area of the triangle
A = ( 0.5 * ( xi*(yj-yk) + xj*(yk-yi) + xk*(yi-yj) ) );

%Calculate alpha ,beta ,omega (required for the B matrix
%and the shape functions)
ali = xj*yk - xk*yj;
alj = xk*yi - xi*yk;
alk = xi*yj - xj*yi;

bei = yj - yk;
bej = yk - yi;
bek = yi - yj;

omi = xk - xj;
omj = xi - xk;
omk = xj - xi;
```

```

%Calculate the shape functions
syms x y;

Si = (1/(2*A)) * (ali + bei*x + omi*y);
Sj = (1/(2*A)) * (alj + bej*x + omj*y);
Sk = (1/(2*A)) * (alk + bek*x + omk*y);

%Calculate the B matrix
B = (1/(2*A)) * [ bei 0 bej 0 bek 0 ; 0 omi 0 omj 0 omk ;...
    omi bei omj bej omk bek ];

%Volume of the triangle
V = A * h;

%Calculate The K matrix
K = V * (B.' * v * B);

disp('Linear triangle K matrix')
disp(K)

```

### Bar structure:

```

clear all;
clc;

syms E A1 A3 ;
% Input parameters
num_elements = 3;
num_nodes = 3;

element(1).nodes = [1 2];
element(2).nodes = [2 3];
element(3).nodes = [3 1];

element(1).csa = A1;
element(2).csa = A3;
element(3).csa = A1;

element(1).E = E;
element(2).E = E;
element(3).E = E;

element(1).length = 1;
element(2).length = sqrt(2);
element(3).length = 1;

element(1).angle = 0;
element(2).angle = 135;
element(3).angle = 90;

for i = 1:num_elements
    element(i).angle = deg2rad(element(i).angle);

```

```

end

% Calculate the stiffness for each element
for i = 1:num_elements
    element(i).k = (element(i).csa * element(i).E ) / element(i).length;
end

% Create the stiffness matrices in the global direction
for i = 1:num_elements
    o = element(i).angle;
    element(i).transf = [ (cos(o))^2 sin(o)*cos(o) -(cos(o))^2 -
sin(o)*cos(o); ...
    sin(o)*cos(o) (sin(o))^2 -sin(o)*cos(o) -(sin(o))^2; ...
    -(cos(o))^2 -sin(o)*cos(o) (cos(o))^2 sin(o)*cos(o); ...
    -sin(o)*cos(o) -(sin(o))^2 sin(o)*cos(o) (sin(o))^2 ];
    [nrows,ncol] = size(element(i).transf);
    for j =1:nrows
        for k= 1:ncol
            if abs(element(i).transf(j,k))<= 1e-3
                element(i).transf(j,k) = 0;
            end
        end
    end
    element(i).Ke = element(i).k * element(i).transf;
    disp(element(i).Ke)
end

%Create the global matrix
K_global = sym(zeros(num_nodes*2,num_nodes*2));

for i = 1:num_elements

    x = element(i).nodes(1);
    y = element(i).nodes(2);
    zx = 2*x-1;
    zy = 2*y-1;
    K_global (zx:zx+1 ,zx:zx+1) = K_global (zx:zx+1 ,zx:zx+1) +
element(i).Ke(1:2 , 1:2);
    K_global (zy:zy+1 ,zy:zy+1) = K_global (zy:zy+1 ,zy:zy+1) +
element(i).Ke(3:4 , 3:4);
    K_global (zx:zx+1 ,zy:zy+1) = K_global (zx:zx+1 ,zy:zy+1) +
element(i).Ke(1:2 , 3:4);
    K_global (zy:zy+1 ,zx:zx+1) = K_global (zy:zy+1 ,zx:zx+1) +
element(i).Ke(3:4 , 1:2);

end

disp('Bar Global K matrix')
pretty(K_global)

```

**The results are as follows:**

**Linear triangle:**

Linear triangle K matrix

```
[ (3*E)/4, E/4, -E/2, -E/4, -E/4, 0]
[ E/4, (3*E)/4, 0, -E/4, -E/4, -E/2]
[ -E/2, 0, E/2, 0, 0, 0]
[ -E/4, -E/4, 0, E/4, E/4, 0]
[ -E/4, -E/4, 0, E/4, E/4, 0]
[ 0, -E/2, 0, 0, 0, E/2]
```

Bar structure:

Bar Global K matrix

```
/ A1 E, 0, -A1 E, 0, 0, 0 \
|
| 0, A1 E, 0, 0, 0, -A1 E |
|
| -A1 E, 0, #1, -#2, -#2, #2 |
|
| 0, 0, -#2, #2, #2, -#2 |
|
| 0, 0, -#2, #2, #2, -#2 |
|
\ 0, -A1 E, #2, -#2, -#2, #1 /
```

where

$$\#1 == A1 E + \#2$$

$$\#2 == \frac{\sqrt{2} A3 E}{4}$$

b)

Both stiffness matrices could not be equivalent; however, values of A1 and A3 could be chosen to make them as close as possible. This could be done by equating the diagonal terms. Hence, this results in the following values:

$$A1 = \frac{3}{4} \text{ and } A3 = \sqrt{2}.$$

c)

Both stiffness matrices are not equal because they represent two different types of analysis hence structures. The linear triangle represents 2D object with a thickness of 1. The bar structure is an assembly of three connected bars, hence an assembly of three 1D elements in a 2D space. The behavior of both structures is expected to be different as the relation between the nodes is defined based on two different modes of analysis.

d)



Setting nu not equal to zero yields the following matrices for plane stress and plane strain cases respectively:

$$E/(4*(\nu^2 - 1)) *$$

ans =

$$\begin{bmatrix} \nu - 3, & -\nu - 1, & 2, & 1 - \nu, & 1 - \nu, & 2*\nu \\ -\nu - 1, & \nu - 3, & 2*\nu, & 1 - \nu, & 1 - \nu, & 2 \\ 2, & 2*\nu, & -2, & 0, & 0, & -2*\nu \\ 1 - \nu, & 1 - \nu, & 0, & \nu - 1, & \nu - 1, & 0 \\ 1 - \nu, & 1 - \nu, & 0, & \nu - 1, & \nu - 1, & 0 \\ 2*\nu, & 2, & -2*\nu, & 0, & 0, & -2 \end{bmatrix}$$

$$E/(4*(2*\nu^2 + \nu - 1)) *$$

ans =

$$\begin{bmatrix} 4*\nu - 3, & -1, & 2 - 2*\nu, & 1 - 2*\nu, & 1 - 2*\nu, & 2*\nu \\ -1, & 4*\nu - 3, & 2*\nu, & 1 - 2*\nu, & 1 - 2*\nu, & 2 - 2*\nu \\ 2 - 2*\nu, & 2*\nu, & 2*\nu - 2, & 0, & 0, & -2*\nu \\ 1 - 2*\nu, & 1 - 2*\nu, & 0, & 2*\nu - 1, & 2*\nu - 1, & 0 \\ 1 - 2*\nu, & 1 - 2*\nu, & 0, & 2*\nu - 1, & 2*\nu - 1, & 0 \\ 2*\nu, & 2 - 2*\nu, & -2*\nu, & 0, & 0, & 2*\nu - 2 \end{bmatrix}$$

The resultant stiffness matrix is different from the one obtained in a). This is due to considering the transverse strain which comes into effect due to Poisson's ratio. It could be noticed that setting nu equal to zero will lead to a). Another observation in the case of plane strain, the K matrix becomes undefined as nu is set to 1/2. This corresponds with the incompressible material case.