Computational Structural Mechanics & Dynamics

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Assignment 3

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Assignment 3.1

$$① \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad ② \quad \mu = G = \frac{E}{2(1+\nu)}$$

① Find E, ν in terms of λ, μ:

$$② \rightarrow (1+\nu) = \frac{E}{2\mu} \xrightarrow{\text{in}①} \lambda = \frac{\cancel{E}\,\nu}{\cancel{E}/_{2\mu} \cdot (1-2\nu)}$$

$$2\mu\nu = \lambda - 2\lambda\nu \Rightarrow \nu = \frac{\lambda}{2(\mu+\lambda)} \quad ③$$

$$② \rightarrow E = 2\mu(1+\nu) = 2\mu\left(1+\frac{\lambda}{2(\mu+\lambda)}\right) \Rightarrow E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad ④$$

② Express Elastic Matrix for plane stress / Strain in terms of λ, μ.

plane stress

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix}$$

New Expressions

$$E_{11} = E_{22} = \frac{E}{1-\nu^2} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \cdot \frac{1}{1-\frac{\lambda^2}{4(\lambda+\mu)^2}} = \frac{\mu(3\lambda+2\mu)}{\cancel{\lambda+\mu}} \cdot \frac{4(\mu+\lambda)^{\cancel{2}}}{4(\lambda+\mu)^2-\lambda^2}$$

$$E_{12} = E_{21} = \frac{E}{1-\nu^2}\nu = \frac{\mu(3\lambda+2\mu)\cdot \cancel{4}^{2}\cancel{(\mu+\lambda)}}{(4(\lambda+\mu)^2-\lambda^2)} \cdot \frac{\lambda}{\cancel{2(\mu+\lambda)}} = \frac{2\mu\lambda(3\lambda+2\mu)}{4(\lambda+\mu)^2-\lambda^2}$$

$$E_{33} = \frac{E}{1-\nu^2} \cdot \frac{1-\nu}{2} = \frac{E}{2(1+\nu)} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \cdot \frac{1}{2\left(1+\frac{\lambda}{2(\mu+\lambda)}\right)}$$

$$= \frac{\mu(3\lambda+2\mu)}{\cancel{\lambda+\mu}} \cdot \frac{\cancel{2}\cancel{(\mu+\lambda)}}{(2\mu+3\lambda)\times\cancel{2}} = \mu$$

$$E_{13} = E_{31} = 0$$

$$E_{23} = E_{32} = 0$$

plane strain:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{bmatrix}$$

$$E_{22} = E_{11} = \frac{E\ (1-\nu)}{(1+\nu)(1-2\nu)} = \frac{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\cdot\left(1-\frac{\lambda}{2(\lambda+\mu)}\right)}{\left(1+\frac{\lambda}{2(\lambda+\mu)}\right)\left(1-\frac{\cancel{2}\lambda}{\cancel{2}(\lambda+\mu)}\right)} =$$

$$= \frac{\frac{\cancel{\mu}\cancel{(3\lambda+2\mu)}}{\cancel{(\lambda+\mu)}}\cdot\frac{(\lambda+2\mu)}{\cancel{2(\lambda+\mu)}}}{\frac{\cancel{(3\lambda+2\mu)}}{\cancel{2(\lambda+\mu)}}\cdot\frac{\cancel{\mu}}{\cancel{(\lambda+\mu)}}} = \lambda+2\mu$$

$$E_{12} = E_{21} = \frac{E\ \nu}{(1+\nu)(1-2\nu)} = \frac{\frac{\cancel{\mu}\cancel{(3\lambda+2\mu)}}{\cancel{(\lambda+\mu)}}\cdot\frac{\lambda}{\cancel{2(\lambda+\mu)}}}{\frac{\cancel{(3\lambda+2\mu)}}{\cancel{2(\lambda+\mu)}}\cdot\frac{\cancel{\mu}}{\cancel{(\lambda+\mu)}}} = \lambda$$

$$E_{33} = \frac{E}{2(1+\nu)} = \frac{\frac{\mu\cancel{(3\lambda+2\mu)}}{\cancel{(\lambda+\mu)}}}{\cancel{2}\left(\frac{\cancel{(3\lambda+2\mu)}}{\cancel{2}\cancel{(\lambda+\mu)}}\right)} = \mu$$

(3) split $E = E_\lambda + E_\mu$

$$E = \begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & \mu\end{bmatrix} = E_\lambda + E_\mu \qquad E_\lambda = \begin{bmatrix}\lambda & \lambda & 0\\ \lambda & \lambda & 0\\ 0 & 0 & 0\end{bmatrix} \qquad E_\mu = \begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$$

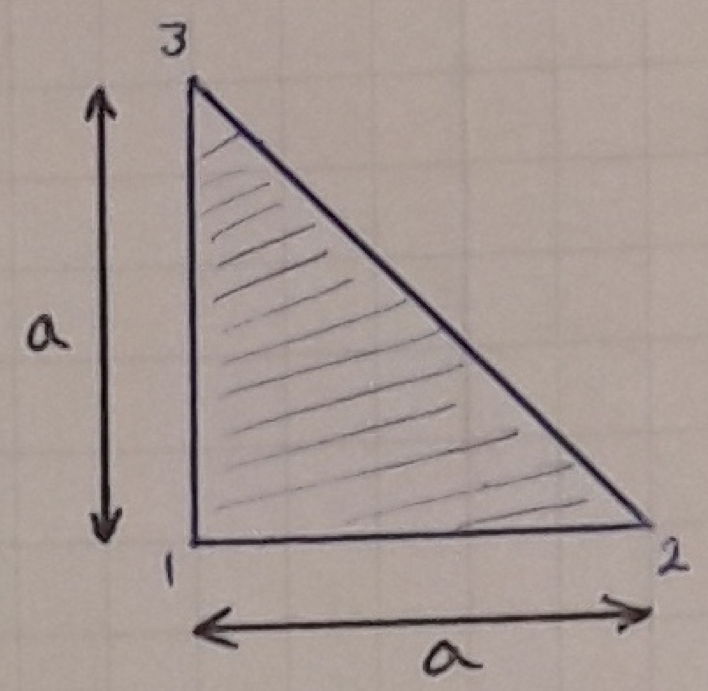
(4) Express E_λ, E_μ in terms of E, ν

$$E_\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$E_\mu = \frac{E}{(1+\nu)}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$$

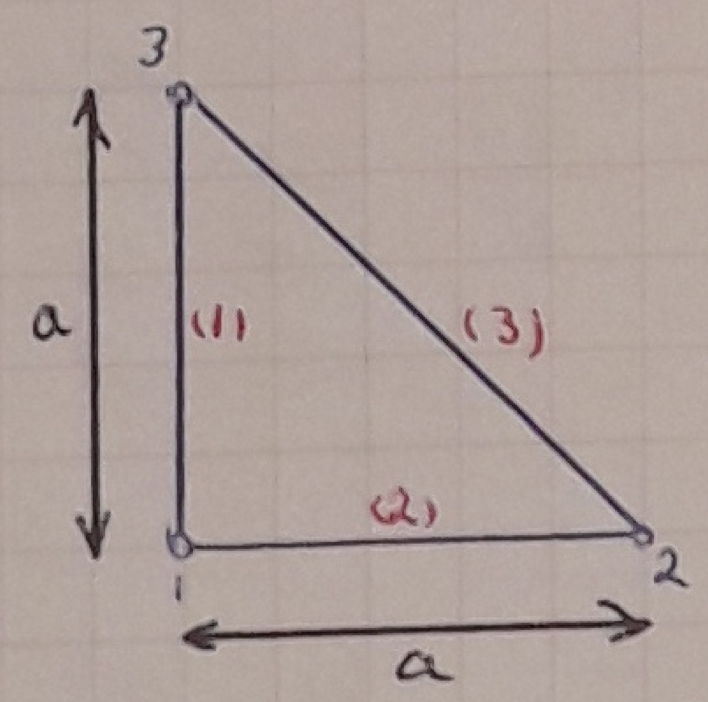
Assignment 3.2

* Compare K_{tri} and K_{bar} :



E , ν

$a = 1$, thickness $= 1$



$A_1 = A_2$, E, $a = 1$

A_3

$$K^{(e)}_{tri} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

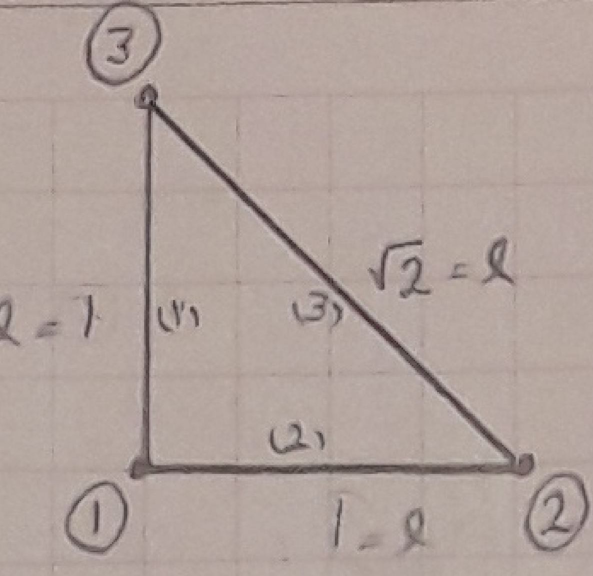
$$E = \frac{E}{1-\nu^2}\begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{vmatrix}$$

$$y_{jk} = y_j - y_k$$

$$K^{(e)}_{tri} = \frac{1}{4 \times \frac{1}{2}}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{6\times3}\begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix}_{3\times3}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}_{3\times6} =$$

$$K^{(e)}_{tri} = \frac{1}{2}\left[\begin{array}{cc|cc|cc} \frac{3E}{2} & \frac{E}{2} & -E & -\frac{E}{2} & -\frac{E}{2} & 0 \\ \frac{E}{2} & \frac{3E}{2} & 0 & -\frac{E}{2} & -\frac{E}{2} & -E \\ \hline -E & 0 & E & 0 & 0 & 0 \\ -\frac{E}{2} & -\frac{E}{2} & 0 & \frac{E}{2} & \frac{E}{2} & 0 \\ \hline -\frac{E}{2} & -\frac{E}{2} & 0 & \frac{E}{2} & \frac{E}{2} & 0 \\ 0 & -E & 0 & 0 & 0 & E \end{array}\right] = E \begin{array}{c} \begin{array}{cccccc} & ① & & ② & & ③ \end{array} \\ \left[\begin{array}{cc|cc|cc} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ \hline -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{array}\right] \end{array} \begin{array}{c} ① \\ \\ ② \\ \\ ③ \\ \end{array}$$

$$K^{(e)}_{bar} = \frac{EA}{L}\begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$



$$K^{(1)}_{bar} = \frac{EA_1}{1}\begin{array}{c} \begin{array}{cc} ① & ③ \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix} \end{array} \begin{array}{c} ① \\ ③ \end{array}$$

$\alpha = 90^\circ \rightarrow S=1$, $\searrow C=0$

$$K^{(2)}_{bar} = \frac{EA_1}{1}\begin{array}{c} \begin{array}{cc} ① & ② \end{array} \\ \begin{bmatrix} +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} ① \\ ② \end{array}$$

$\alpha = 0 \rightarrow S=0$, $\searrow C=1$

$$K^{(3)}_{bar} = \frac{EA_3}{\sqrt{2}} \times \frac{1}{2}\begin{array}{c} \begin{array}{cc} ② & ③ \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} ② \\ ③ \end{array}$$

$\alpha = 135 \rightarrow C = -\frac{1}{\sqrt{2}}$, $\searrow S = \frac{1}{\sqrt{2}}$

$$K_{bar} = E \begin{array}{c} \begin{array}{ccc} ① & ② & ③ \end{array} \\ \begin{bmatrix} A_1 & 0 & -A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & -A_1 \\ -A_1 & 0 & B+A_1 & -B & -B & B \\ 0 & 0 & -B & B & B & -B \\ 0 & 0 & -B & B & B & -B \\ 0 & -A_1 & B & -B & -B & B+A_1 \end{bmatrix} \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

$\left(B = \frac{A_3}{2\sqrt{2}} \right)$

2) is there any set of $A_1 = A_2$ and A_3 to make them equal or similar?

* These two Matrix have completely different components and in No-way Can be equally the Same.

* But in order to make them as similar as possible we could make diagonal components the same :

$E_{11} = A_1 = \frac{3}{4}$

$E_{33} = B + A_1 = \frac{1}{2} \longrightarrow B = \frac{-1}{4} \Rightarrow A_3 = -\frac{1}{\sqrt{2}}$ $\qquad E_{44} = B = \frac{1}{4} \Rightarrow A_3 = \frac{1}{\sqrt{2}}$

3) physical explanation :

* These two stiffness matrices are not equal, and if have physical explanations in Bar Model there is no Mass in the middle and all stiffness are concentrated on bars, but in Triangular Model, stiffness matrix includes all Elasticity terms

4. Considering $\nu \neq 0$ extract some conclusion.

$$K^{(e)}_{Tri} = \frac{hE}{4A(1-\nu^2)} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

* The main difference here with the case of ($\nu = 0$) is that when $\nu \neq 0$ there would be less Zero terms in the global stiffness Matrix, so one could expect a more stiff behavior from this matrix compared to the previous model (i.e: $\nu = 0$)

* on the other hand there remains a huge difference between K_{Tri} and K_{bar} and as it was discussed previously it is mainly to the lack od stiff elements in the middle part od model in bar representation.