Computational Structural Mechanics & Dynamics

M. Mohsen Zadeh kamand

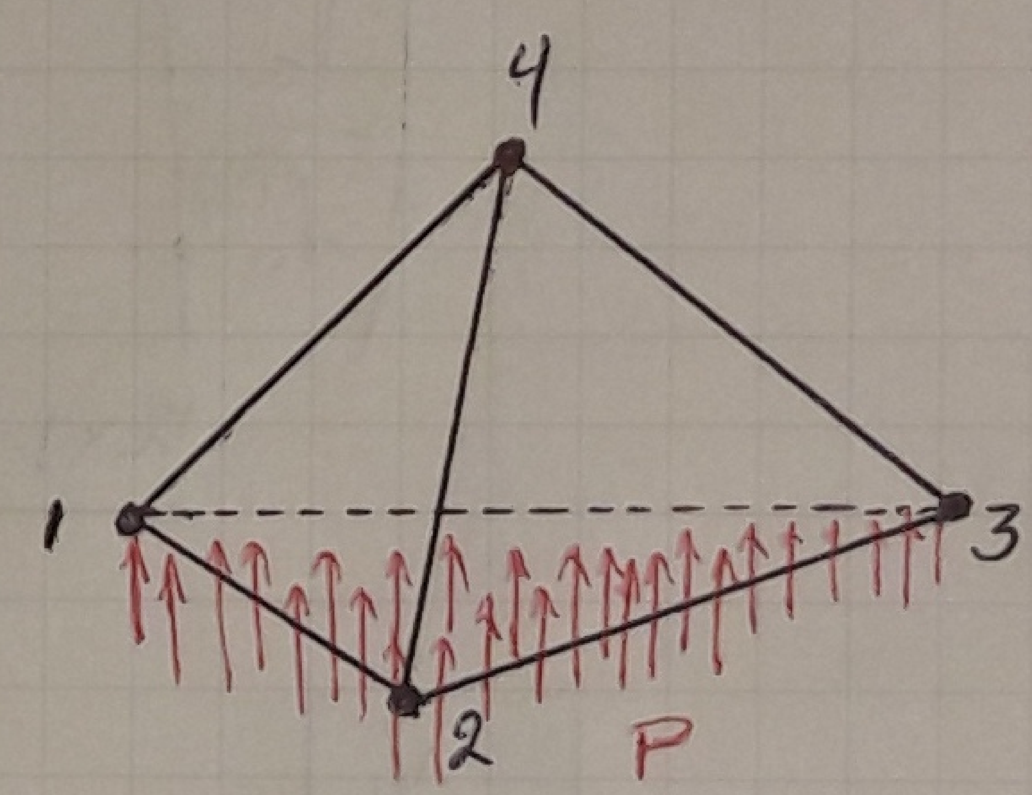
M.Sc. Computational Mechanics (2016-2018)

Assignment 5

(13 March 2017)

(Assignment 5.1) Solid Elements

- Normal pressure acting on face 1-2-3
- compute f^e

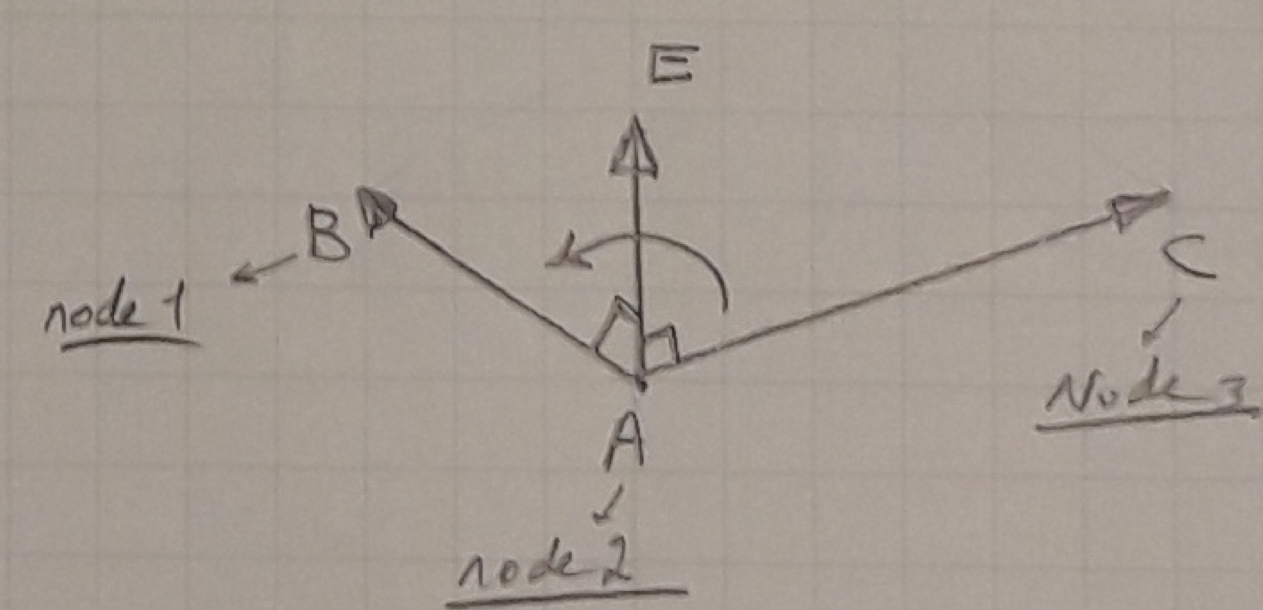


$$|F_t| = P \cdot A$$

$|F_t|$: Traction force; P: pressure Normal to face; A: Area of face

$\underline{P}$ is Normal to face 1-2-3, So we only need to know $\vec{n}$ direction

$$\vec{V} = \overrightarrow{AC} \times \overrightarrow{AB} \qquad \vec{n} = \frac{\vec{V}}{|\vec{V}|}$$



$$\vec{V} = \begin{vmatrix} i & j & k \\ AC_x & AC_y & AC_z \\ AB_x & AB_y & AB_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ x_{32} & y_{32} & z_{32} \\ x_{12} & y_{12} & z_{12} \end{vmatrix}$$

$$y_{ij} = y_i - y_j$$

$$\vec{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} y_{32} z_{12} - y_{12} z_{32} \\ z_{32} x_{12} - x_{32} z_{12} \\ x_{32} y_{12} - y_{32} x_{12} \end{bmatrix} = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$$

$$\vec{n} = \frac{\vec{V}}{|\vec{V}|}$$

$\vec{n}$: Normal face vector (unit)

$$A_4 = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$A_4 = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)$$

$$f^{(4)} = \frac{1}{3} \times P \times A_4 \begin{pmatrix} f_1^{(4)} \\ \hline f_2^{(4)} \\ \hline f_3^{(4)} \\ \hline f_4^{(4)} \end{pmatrix}$$

$$f_i^{(4)} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

* Second Solution way:

$$\left\{ f_p^{(4)} \right\} = \int_S N^T \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix} dS \qquad N^T = \left(\begin{array}{ccc|ccc|ccc|ccc} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{array} \right)$$

$\underline{3 \times 12}$

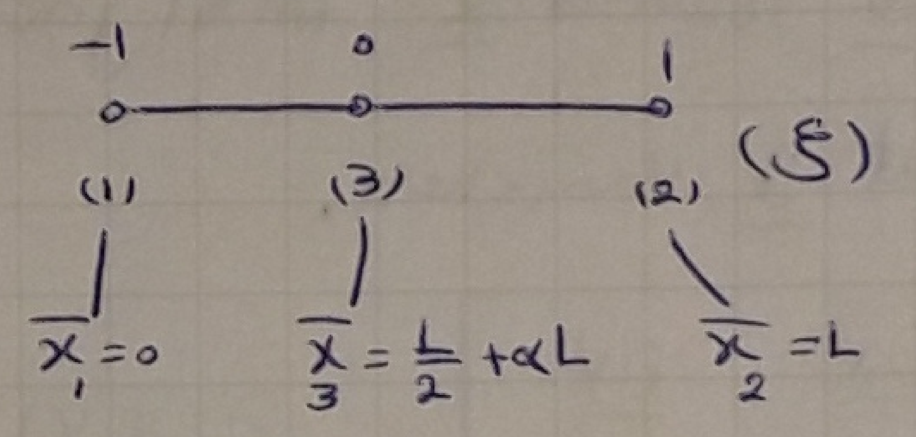
$$\left\{ f_P^{(4)} \right\} = \begin{pmatrix} f_1^{(4)} \\ \hline f_2^{(4)} \\ \hline f_3^{(4)} \\ \hline f_4^{(4)} \end{pmatrix} \qquad f_i^{(4)} = \begin{pmatrix} f_{1x}^{(4)} \\ f_{1y}^{(4)} \\ f_{1z}^{(4)} \end{pmatrix}$$

$\underline{12 \times 1}$ $\qquad$ $\underline{3 \times 1}$

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \frac{1}{6V} \begin{bmatrix} 6V_1 & a_1 & b_1 & c_1 \\ 6V_2 & a_2 & b_2 & c_2 \\ 6V_3 & a_3 & b_3 & c_3 \\ 6V_4 & a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$$

Assignment 5.2 (FEM Convergence requirement)

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$



show for $J=0$ $\quad |\alpha|_{min} = \pm\frac{1}{4}$

$$\begin{cases} N_1(\xi) = \frac{1}{2}(\xi - 0)(\xi - 1) \\ N_2(\xi) = \frac{1}{2}(\xi - 0)(\xi + 1) \\ N_3(\xi) = (-1)(\xi - 1)(\xi + 1) \end{cases}$$

$J = \dfrac{dx}{d\xi}$ $\Rightarrow$ Need x $\qquad x = \sum_{i=1}^{3} N_i^e x_i$

$$x = \cancel{\tfrac{1}{2}\xi(\xi-1)(0)} + \tfrac{1}{2}\xi(\xi+1)(L) + (1-\xi)(1+\xi)\left(\tfrac{L}{2}+\alpha L\right)$$

$$x = \xi^2\left(\cancel{\tfrac{L}{2}} - \cancel{\tfrac{L}{2}} - \alpha L\right) + \xi\left(\tfrac{L}{2}\right) + \left(\tfrac{L}{2}+\alpha L\right)$$

$$J = \frac{dx}{d\xi} = \frac{d\left[\xi^2(-\alpha L) + \xi\left(\frac{L}{2}\right) + \left(\frac{L}{2}+\alpha L\right)\right]}{d\xi} = -2\alpha L \xi + \frac{L}{2}$$

if $J = 0 \Rightarrow \xi = \dfrac{1}{4\alpha} \longrightarrow -1 \leq \xi \leq 1 \Rightarrow \alpha \geq \dfrac{1}{4}$ or $\alpha \leq -\dfrac{1}{4}$

$$\Rightarrow |\alpha|_{min} = \frac{1}{4}$$

show strain becomes infinite at an end point (Singularity Interpretation)

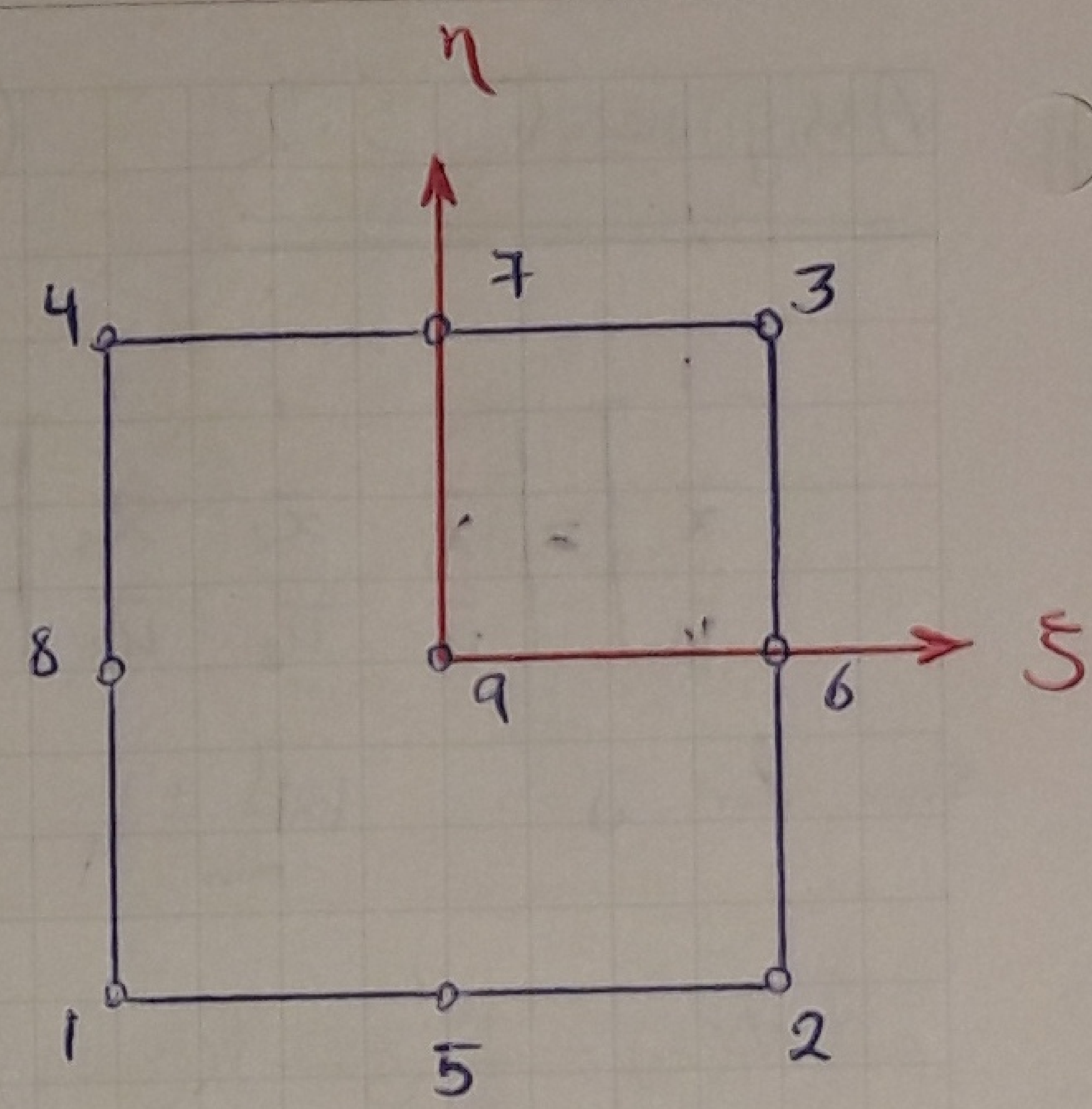
$$e = \frac{du}{dx} \qquad u = \sum_{i=1}^{3} N_i U_i \Rightarrow \frac{dU}{dx} = \sum_{i=1}^{3} \frac{dN_i}{dx} U_i = \sum_{i=1}^{3} \frac{dN_i}{d\xi}\cdot\frac{d\xi}{dx} U_i$$

$$= \sum_{i=1}^{3} \frac{dN_i}{d\xi}\cdot J^{-1} U_i \longrightarrow \infty$$

if $J = 0 \Rightarrow J^{-1} \rightarrow \infty$

if jacobian vanishes ($J=0$), then strain becomes infinit (physical scheme of singularity!)

Assignment 5.3



Move Node 5 toward 2 until jacobian determinant at 2 vanishes.

$$|J| = \det \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{\partial(x,y)}{\partial(\xi,\eta)}$$

need expression of x & y in terms of ξ, η

$$x = \sum_{i=1}^{9} N_i\, x_i \qquad y = \sum_{i=1}^{9} N_i\, y_i$$

Node i	N_i	$\frac{dN_i}{d\xi}$	$\frac{dN_i}{d\eta}$
1	$\frac{1}{4}\xi(\xi-1)\eta(\eta-1)$	$\frac{1}{4}(2\xi-1)\eta(\eta-1)$	$\frac{1}{4}\xi(\xi-1)(2\eta-1)$
2	$\frac{1}{4}\xi(\xi+1)\eta(\eta-1)$	$\frac{1}{4}(2\xi+1)\eta(\eta-1)$	$\frac{1}{4}\xi(\xi+1)(2\eta-1)$
3	$\frac{1}{4}\xi(\xi+1)\eta(\eta+1)$	$\frac{1}{4}(2\xi+1)\eta(\eta+1)$	$\frac{1}{4}\xi(\xi+1)(2\eta+1)$
4	$\frac{1}{4}\xi(\xi-1)\eta(\eta+1)$	$\frac{1}{4}(2\xi-1)\eta(\eta+1)$	$\frac{1}{4}\xi(\xi-1)(2\eta+1)$
5	$-\frac{1}{2}(\xi-1)(\xi+1)\eta(\eta-1)$	$-\xi\eta(\eta-1)$	$\frac{1}{2}(1-\xi^2)(2\eta-1)$
6	$+\frac{1}{2}\xi(\xi+1)(1-\eta)(1+\eta)$	$\frac{1}{2}(2\xi+1)(1-\eta^2)$	$-\xi(\xi+1)\eta$
7	$-\frac{1}{2}(\xi-1)(\xi+1)\eta(\eta+1)$	$-\xi\eta(\eta+1)$	$\frac{1}{2}(1-\xi^2)(2\eta+1)$
8	$+\frac{1}{2}\xi(\xi-1)(1-\eta)(1+\eta)$	$\frac{1}{2}(2\xi-1)(1-\eta^2)$	$-\xi(\xi-1)\eta$
9	$(\xi-1)(\xi+1)(\eta-1)(\eta+1)$	$-2\xi(1-\eta^2)$	$(1-\xi^2)(-2\eta)$

i	X_i	Y_i
1	0	0
2	L	0
3	L	L
4	0	L
5	$\frac{L}{2} + \alpha L$	0
6	L	L/2
7	L/2	L
8	0	L/2
9	L/2	L/2

$$\frac{dx}{d\xi} = \sum_{i=1}^{9} \frac{dN_i}{d\xi} x_i = 0 + (L)\left(\frac{1}{4}(2\xi+1)\eta(\eta-1)\right) + L\left(\frac{1}{4}(2\xi+1)\eta(\eta+1)\right) + 0 +$$

$$+\left(\frac{L}{2}+\alpha L\right)\left(-\xi\eta(\eta-1)\right) + (L)\left(\frac{1}{2}(2\xi+1)(1-\eta^2)\right) + \left(\frac{L}{2}\right)\left(-\xi\eta(\eta+1)\right)$$

$$+ 0 + \left(\frac{L}{2}\right)\left(-2\xi(1-\eta^2)\right) =$$

$$= \frac{L}{2}\left(1 + 2\alpha\xi\eta - 2\alpha\xi\eta^2\right)$$

$$\frac{dx}{d\eta} = \sum_{i=1}^{9} \frac{dN_i}{d\eta} x_i = 0 + (L)\left(\frac{1}{4}(2\eta-1)\xi(\xi+1)\right) + L\left(\frac{1}{4}(2\eta+1)\xi(\xi+1)\right) + 0 +$$

$$+\left(\frac{L}{2}+\alpha L\right)\left(\frac{1}{2}(1-\xi^2)(2\eta-1)\right) + L\left(-\xi\eta(\xi+1)\right) +$$

$$+\left(\frac{L}{2}\right)\left(\frac{1}{2}(1-\xi^2)(2\eta+1)\right) + 0 + \frac{L}{2}(-2\eta)(1-\xi^2) =$$

$$= L\left(\frac{\alpha}{2}(1-\xi^2)(2\eta-1)\right)$$

$$\frac{dy}{d\xi} = \sum_{i=1}^{9} \frac{dN_i}{d\xi} y_i = 0 + 0 + L\left(\frac{1}{4}(2\xi+1)\eta(\eta+1)\right) + L\left(\frac{1}{4}(2\xi-1)\eta(\eta+1)\right) + 0$$

$$+\frac{L}{2}\left(\frac{1}{2}(2\xi+1)(1-\eta^2)\right) + L\left(-\xi\eta(\eta+1)\right) +$$

$$+\frac{L}{2}\left(\frac{1}{2}(2\xi-1)(1-\eta^2)\right) + \frac{L}{2}\left(-2\xi(1-\eta^2)\right) =$$

$$= 0$$

$$\frac{dy}{d\eta} = \sum_{i=1}^{9} \frac{dN_i}{d\eta} y_i = 0 + 0 + L\left(\frac{1}{4}\xi(\xi+1)(2\eta+1)\right) + L\left(\frac{1}{4}\xi(\xi-1)(2\eta+1)\right) + 0$$

$$+\frac{L}{2}\left(-\xi(\xi+1)\eta\right) + L\left(\frac{1}{2}(1-\xi^2)(2\eta+1)\right) +$$

$$+\frac{L}{2}\left(-\xi(\xi-1)\eta\right) + \frac{L}{2}\left((1-\xi^2)(-2\eta)\right) =$$

$$= \frac{L}{2}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{vmatrix} = \frac{L^2}{4}\left(1+2\alpha\xi\eta - 2\alpha\xi\eta^2\right) \overset{\text{if}}{=} 0 \Rightarrow \boxed{\alpha = \frac{1}{2\xi\eta(\eta-1)}}$$

if Node 5 moves toward 2 then using $\left|\begin{matrix}\xi=1\\ \eta=-1\end{matrix}\right.$ (for 2) we have $\alpha = \frac{1}{4}$

so if we move one middle point toward a corner node in $|\alpha| = \frac{1}{4}$. determinant of Jacobian Vanishes.